1. **AP,GP, series (summation), MOD, MI, partial fractions**

- be aware of formulas for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ $\sum_{r=1}^{n} r^3$ and correction measures to be taken when lower limit r is not equals to 1.

- be aware of formulas for nth term and sum to n terms of an AP as well as a GP.

-ability to appreciate the concept of sum to infinity for a GP with criteria that |r|<1

- recognise **common usage of GP** in compound interest/banking problems

-ability to **make relation between terms of an AP and GP** (eg 1st, 6th and 11th terms of an AP are also the 3rd, 2nd and 1st terms of a GP)

-ability to identify a need for **method of difference** when required in question. (Note that MOD is usually integrated within a question which tests the concept of summation of series)

-ability to generate full answer layout for mathematical induction (opening header, P_1, P_k, P_{k+1} and concluding statement)

-ability to break down improper polynomial representations via long division (synthetic division) and subsequently generating its constituent polynomial factors

PREDICTED QUESTION STRUCTURES :

a. A descending geometric series has first term 1 and the common ratio r is positive. The sum of the first 5 terms is twice the sum of terms from the 6th to 15th inclusive. Prove that $r^5 = \frac{1}{2}(\sqrt{3} - 1)$.

*b. The product of 3 numbers, which form a geometric progression, is 216. If the third number is decreased by 3, the three numbers now form an arithmetic progression. Find the original numbers.

c. .(i) Express $\frac{r}{(r+1)(r+2)(r+3)}$ in partial fractions.

(ii) Show that $\sum_{r=1}^{n} \frac{r}{(r+1)(r+2)(r+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)}$.

(iii) Hence find the value of $\lim_{n\to\infty}\sum_{r=1}^{n} \frac{r}{(r+1)(r+2)(r+3)}$.

d. Evaluate $\sum_{r=n+1}^{2n} r^2 + 3k + r$

e. Use induction to prove that $\sum_{r=2}^{n} (r^2 + r + 1)r! = (n+1)^2 n! - 4.$

*f. At the end of a month, a customer owes a bank $1500. In the middle of the month, the customer pays $x to the bank where x<1000, and at the end of the month the bank adds interest at a rate of 4% of the remaining amount still owed. This process continues every month until the money owed is repaid in full.

(i) Find the value of x for which the customer still owes $1500 at the start of every month.

(ii) Find the value of x for which the whole amount owed is paid off exactly after the second payment.

(iii) Show that the value of x for which the whole amount owed is paid off exactly after the (n+1)th payment is given by

$$x = \frac{1500r^n(r-1)}{r^{n+1} - 1}$$, where r=1.04

*g The sum of the first twenty terms of an arithmetic series is 610. The first, third and eleventh terms of this series are also the third, second and first term of a geometric series. Find the first term of the geometric series. Find the sum of the first n terms of the geometric series and find the sum to infinity of the series.

SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

b. Let the three numbers be ar^{-1}, a, ar

$$\therefore\ (ar^{-1})(a)(ar) = 216 \quad \text{ie} \quad a^3 = 216 \Rightarrow a = 6$$

Also, $ar - 3 - a = a - ar^{-1}$ Since $a = 6$,

$$6r - 15 = -6r^{-1} \Rightarrow 2r - 5 + 2r^{-1} = 0 \Rightarrow 2r^2 - 5r + 2 = 0$$

Solving gives $r = \frac{1}{2}$ or $r = 2$

Hence, the 3 numbers are **3, 6, 12** (for $r = 2$) or **12, 6, 3** (for $r = \frac{1}{2}$) (shown)

f. (i) 1500=(1500-x)(1.04) $\Rightarrow$ $x = \$57.69$ (shown)

(ii) After 1st payment of $x, the amount owed= (1500-x)(1.04)

Therefore, (1500-x)(1.04)=x $\Rightarrow$ $x = \$764.71$ (shown)

(iii) After the second payment of \$x, amount owed at beginning of 3rd month is

$[(1500-x)1.04 - x](1.04) = 1500(1.04)^2 - 1.04^2 x - 1.04x$

After the second payment of \$x, amount owed at beginning of 4th month is

$[1500(1.04)^2 - 1.04^2 x - 1.04x - x](1.04)$

$= 1500(1.04)^3 - 1.04^3 x - 1.04^2 x - 1.04x$

After nth payment of \$x, the amount still owed at the beginning of the (n+1)th month

$= 1500(1.04)^n - 1.04^n x - 1.04^{n-1} x - \ldots\ldots\ldots\ldots - 1.04^2 x - 1.04x$

$= 1500r^n - x(r + r^2 + \ldots\ldots\ldots + r^n)$ where r=1.04

At the (n+1)th payment,

$x = 1500r^n - x(r + r^2 + \ldots\ldots\ldots + r^n)$

$x(1 + r + r^2 + \ldots\ldots + r^n) = 1500r^n$

$\Rightarrow x\left(\dfrac{r^{n+1} - 1}{r - 1}\right) = 1500r^n \Rightarrow x = \dfrac{1500r^n(r-1)}{r^{n+1} - 1}$ (shown)

g. $\dfrac{20}{2}[2a + (20-1)d] = 610 \Rightarrow 20a + 190d = 610$ -------------(1)

Since the first, third and eleventh term of the AP forms a GP,

then $\dfrac{a}{a + 2d} = \dfrac{a + 2d}{a + 10d} \Rightarrow a^2 + 10ad = a^2 + 4ad + 4d^2$

Simplifying gives $3a = 2d$ -------------------------------------(2)

Solving (1) and (2) gives $a = 2, d = 3$

$\therefore$ first term of GP=eleventh term of AP=2+10(3)=32 (shown)

common ration $r = \dfrac{a}{a + 2d} = \dfrac{2}{2 + 6} = \dfrac{1}{4}$ (shown)

$$S_n = \frac{32\left[1 - \left(\frac{1}{4}\right)^n\right]}{1 - \frac{1}{4}} = \frac{128}{3}\left[1 - \left(\frac{1}{4}\right)^n\right]$$ (shown)

$$S_\infty = \frac{a}{1-r} = \frac{32}{1-\frac{1}{4}} = \frac{128}{3} \qquad \text{(shown)}$$

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2. **Binomial series expansion**

-be aware of basic structural expansion of a binomial series
(Note that the term independent of x in the original compressed structure $(k + ax)^n$ must be maintained at a value of 1, ie k=1)

-ability to **compute the coefficient of the rth term** within the series

-ability to consolidate coefficients of various individual terms through multiplication with other series based on question requirements
eg finding series up to and including term in x^5 based on $(1-x)^{-3}(3+x^2+4x^4)$

-ability to use binomial series to **make suitable approximations**

-ability to obtain range of values of x for which the expansion is valid.

PREDICTED QUESTION STRUCTURES :

*a. Given the first three terms in the expansion of $(1-ax)^{-b}$ in ascending powers of x are $1+6x+24x^2+\ldots..$, where a and b are positive constants, find the values of a and b. Show that the coefficient of x^n is $(n+1)(n+2)2^{n-1}$ for n=0,1,2.

b. Find the coefficient of x^r in the series $(1-x)^{-3}$.

*c. Expand $(\frac{1-x}{1+x})^n$ in ascending powers of x up to and including the term in x^2. State the set of values of x for which the expansion is valid. Hence find an approximation to the fourth root of $\frac{19}{21}$ in the form $\frac{p}{q}$, where p and q are positive integers.

SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

a. $(1-ax)^{-b} = 1+(ab)x+\frac{(-b)(-b-2)}{2!}(-ax)^2+\ldots\ldots\ldots$

By comparison, $ab=6$ and $\frac{b^2+b}{2}(a)^2 = 24$

Hence, $\frac{b^2+b}{2}(\frac{36}{b^2}) = 24$

$18(b+b^2) = 24b^2$

$$3b^2 + 3b = 4b^2 \Rightarrow b^2 - 3b = 0 \quad \therefore b = 3, a = 2 \text{ (shown)}$$

Coefficient of x^n is $= \dfrac{(-b)(-b-1)(-b-2)................(-b-n+1)}{n!}(-a)^n$

$$= \frac{(-3)(-4)(-5)...........(-n+2)}{n!}(-a)^n \qquad (b=3)$$

$$= \frac{(-1)^n(3)(4)(5)...........(n+2)}{n!}(-1)^n a^n$$

$$= \frac{(n+2)!}{2n!}a^n = (n+1)(n+2)2^{n-1} \text{ (shown)} \qquad (a=2)$$

c.

$$\left(\frac{1-x}{1+x}\right)^n = (1-x)^n(1+x)^{-n} = \left[1-nx+\frac{(n)(n-1)}{2}(-x)^2+..........\right]\left[1-nx+\frac{(-n)(-n-1)}{2}(x)^2+..........\right]$$

$$= \left[1-nx+\left(\frac{n^2-n}{2}\right)x^2+..........\right]\left[1-nx+\left(\frac{n+n^2}{2}\right)x^2+..........\right]$$

$$= 1-nx+\left(\frac{n+n^2}{2}\right)x^2-nx+n^2x^2+\left(\frac{n^2-n}{2}\right)x^2+...............$$

$$\approx 1-2nx+2n^2x^2 \text{ (shown)}$$

Expansion is valid for $|x|<1$.

Let $x = \dfrac{1}{20}$, $n = \dfrac{1}{4}$,

then $\left(\dfrac{1-x}{1+x}\right)^n = \left(\dfrac{1-\frac{1}{20}}{1+\frac{1}{20}}\right)^{\frac{1}{4}} = \left(\dfrac{19}{21}\right)^{\frac{1}{4}} = 1-2\left(\dfrac{1}{4}\right)\left(\dfrac{1}{20}\right)+2\left[\left(\dfrac{1}{4}\right)\left(\dfrac{1}{20}\right)\right]^2 = \dfrac{3121}{3200}$ (shown)

3. Graphing

-ability to obtain equations of asymptotes and turning points of graphs via calculation
-ability to do transformation of graphs the forms **af(x)+b and/or f(ax+b)**

-ability to discern **qualitatively** the physical meaning of transformations, eg scaling of graph parallel to y axis, translation of graph along x axis etc

-ability to appropriately produce graphs of **parabolas, hyperbolas, ellipses and circles.**

-ability to obtain the graphs of $f(|x|)$, $|f(x)|$, $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = f'(x)$

-ability to deduce original graph based on presentation of series of modified graphs

PREDICTED QUESTION STRUCTURES :

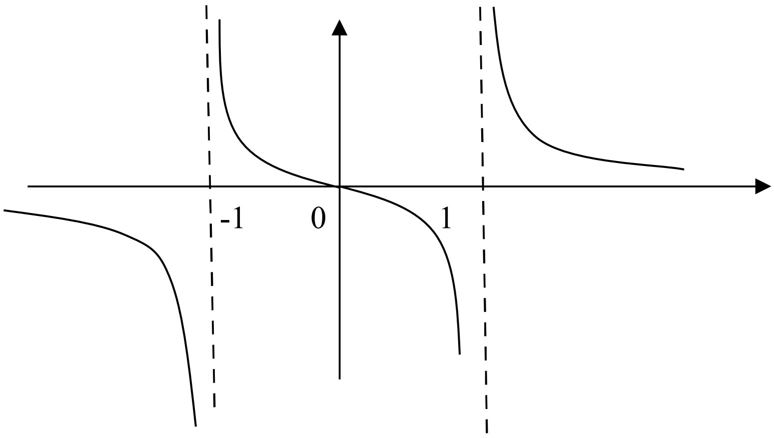
a. The diagram shows the graph of y=f(x). On separate diagrams, sketch the graph of

(i) $y = |f(x)|$

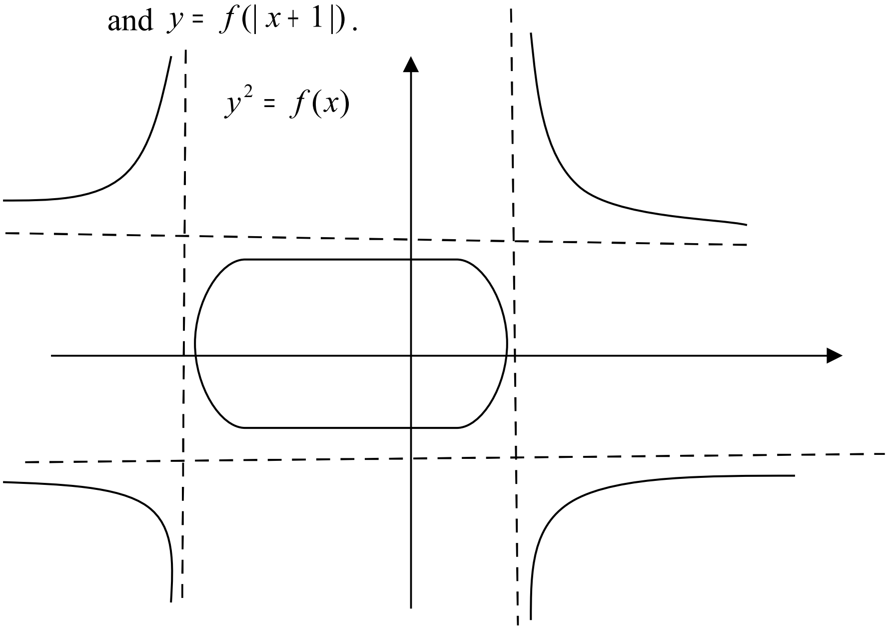
(ii) $y = f(|x|)$

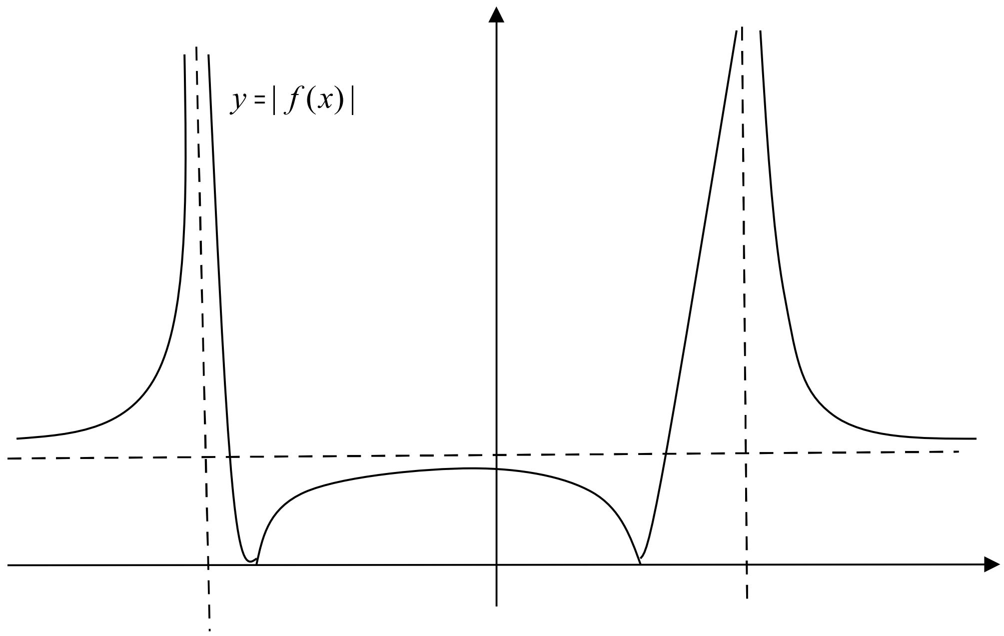
(iii) $y^2 = f(x)$

(iv) $y = \frac{1}{f(x)}$



b. The sketches below show the graphs of $y^2 = f(x)$ and $y = |f(x)|$ for a certain function f. Deduce the graph of y=f(x). Draw a sketch of the curves $y = f(|x|)$ and $y = f(|x+1|)$.





4. Differentiation (applications of differentiation)

-ability to differentiate directly wrt to the variable contained, functions to be differentiated include algebraic, trigonometric, exponential (possibly with a combination of various function types) and techniques such a **product/quotient rules,** implicit differentiation will be employed.

-ability to generate a stipulated nth order differential equation based on question requirements (see predicted question structures b)

-ability to produce steps to find derivatives of basic functions, eg prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

-ability to solve for equations of tangents/normals to a given curve in any fashion stipulated by the question including **parametric representations** of curves

-appreciate the utilisation of differentiation in the context of **geometrical minimisation/maximisation problems**, eg maximum volume of expanding cylinder within sphere as well as rates of changes of parameters in geometrical structures, eg rate of change of radius

PREDICTED QUESTION STRUCTURES :

a. Find $\frac{dy}{dx}$ if

* (i) $\tan y = x \tan^{-1} x$ (ii) x+y+sin(xy)= 2 (iii) $x \cos y + x^3 = -\tan^{-1} y$

b. . If $y = e^x \ln x$, (i) Find $\frac{dy}{dx}$.

(ii) Show that $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - (1+x)y = 2e^x$

*c. An inverted cone of base radius 4cm and height 8cm is initially filled with water. Water drips out from the vertex at a rate of $2\pi \quad cm^3 s^{-1}$. Find the rate of decrease in the depth of the water in the cone 16 seconds after the dripping starts.

d. (i) Given that $x^2 - 2xy + 2y^2 = 4$, find an expression for $\frac{dy}{dx}$ in terms of x and y.

Find the coordinates of each point on the curve $x^2 - 2xy + 2y^2 = 4$ at which the tangent is parallel to the x axis.

*(ii) A curve is defined by the parametric equations $x = t^2, y = t^3$. Show that the equation of the tangent to the curve at the point P (p^2, p^3) is

$2y - 3px + p^3 = 0$. Show that there is just one point on the curve at which the tangent passes through the point (-3,-5), and determine the coordinates of this point.

*e. A length of channel of given depth d is to be made from a rectangular sheet of metal of width 2a. The metal is to be bent in such a way that the cross section ABCD is as shown in the figure, with AB+BC+CD=2a and with AB and CD inclined to the line BC at an angle θ .

Show that BC=2($a - d\cos ec\theta$) and that the area of the cross section ABCD is $2ad + d^2(\cot\theta - 2\cos ec\theta)$.

Show that the maximum value of $2ad + d^2(\cot\theta - 2\cos ec\theta)$, as θ varies, is $d(2a - d\sqrt{3})$.

By considering the length of BC, show that the cross sectional area can only be made equal to this maximum value if $2d \le a\sqrt{3}$.

*f . Show that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

a (i) $\tan y = x\tan^{-1}x$ Differentiating both sides $wrt\ \ x$,

$$\sec^2 y\frac{dy}{dx} = \tan^{-1}x + x\left(\frac{1}{1+x^2}\right) = \frac{\left(1+x^2\right)\tan^{-1}x + x}{1+x^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{\left(1+x^2\right)\tan^{-1}x + x}{\left(1+x^2\right)\left(\sec^2 y\right)} = \frac{\left(1+x^2\right)\tan^{-1}x + x}{\left(1+x^2\right)\left(1+\tan^2 y\right)} = \frac{\left(1+x^2\right)\tan^{-1}x + x}{\left(1+x^2\right)\left[1+\left(x\tan^{-1}x\right)^2\right]}$$

c. $\frac{r}{h} = \frac{4}{8} = \frac{1}{2} \Rightarrow\ r = \frac{1}{2}h$

At any time t, volume remaining V=$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2(h) = \frac{1}{12}\pi h^3$

Differentiating V with respect to h, $\frac{dV}{dh} = \frac{\pi}{4}h^2$

When t =16, $\frac{1}{12}\pi h^3 = \frac{1}{3}\pi\,(4)^2(8) - 2(16) \Rightarrow\ h^3 = 128$

By the chain rule, $\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt} \Rightarrow\ -2\pi = \frac{\pi}{4}(\sqrt[3]{128})^2\times\frac{dh}{dt}$

$$\therefore\ \frac{dh}{dt} = -0.315cms^{-1}\ \text{(shown)}$$

d(ii) $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$

At P, $\frac{dy}{dx} = \frac{3}{2}p$

Equation of tangent is $y - p^3 = \frac{3}{2}p\,(x - p^2)$

Rearranging gives $2y - 3px + p^3 = 0$ (shown)

Since it passes through (-3,-5),

2(-5)-3p (-3)+p^3=0$\Rightarrow\ -10 + 9p + p^3 = 0$

$(p-1)(p^2 + p + 10) = 0$

$\therefore\ p = 1$ and point is $(1^2,\ 1^3) = (1, 1)$ (shown)

(Quadratic polynomial has no real roots since determinant $= b^2 - 4ac = -39 < 0$) (shown)

e. $\sin\theta = \frac{d}{AB} \Rightarrow\ AB = CD = d\cos ec\theta$

$BC = 2a - AB - CD = 2a - 2d\cos ec\theta = 2(a - d\cos ec\theta)$ (shown)

Area of cross section $ABCD$

$= 2d(a - d\cos ec\theta) + 2\left[\frac{1}{2}(d)(d\cot\theta)\right] = 2d(a - d\cos ec\theta) + d(d\cot\theta)$

$= 2ad + d^2(\cot\theta - 2\cos ec\theta)$ (shown)

Let $y = 2ad + d^2(\cot\theta - 2\cos ec\theta)$

$$\frac{dy}{d\theta} = d^2\left[-\operatorname{cosec}^2\theta + 2\cot\theta\operatorname{cosec}\theta\right]$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \operatorname{cosec}^2\theta = 2\cot\theta\operatorname{cosec}\theta \rightarrow \operatorname{cosec}\theta = 2\cot\theta$$

$$\therefore \cos\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}$$

Maximum value = $2ad + d^2\left[\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right] = d(2a - d\sqrt{3})$ (shown)

$BC = 2(a - d\operatorname{cosec}\theta)$, putting in $\theta = \frac{\pi}{3}$,

$$BC = 2\left(a - \frac{2}{\sqrt{3}}d\right)$$

Since $BC \geq 0$, $a \geq \frac{2}{\sqrt{3}}d \Rightarrow 2d \leq \sqrt{3}a$ (shown)

f. Let $y = \tan^{-1}x$, then $\tan y = x$

Differentiating both sides wrt x gives

$$\sec^2 y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \text{ (shown)}$$

(Similar approaches shall be taken for proving the derivatives of $\sin^{-1}x$ and $\cos^{-1}x$)

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5. **Differential equations/Maclaurin's Series**

-ability to solve **first and second order** variables separable differential equations

-ability to transform a differential equation into a viable form for resolution by **means of a substitutio**n given in the question

- ability to construct a **particular solution** for a given differential equation based on certain given conditional inputs (eg when t=0, v=5m/s)

-ability to **sketch a family of solution curves** for the differential equation solved

-ability to generate a Maclaurin's series of a given function through repeated processes of differentiation to obtain the coefficients of the series itself

-ability to utilise the Maclaurin's series obtained (typically derived in first part of question) for various interpretations, including

(i) deducing the series expansion for other functions/expressions (note that such methods could involve **integration/differentiation) ,**

(ii) to produce suitable approximations (eg use the series expansion to deduce the value of $\sin(\frac{\pi}{12})$ via substituting a suitable value of x within the series)

PREDICTED QUESTION STRUCTURES :

a. It is given that $\ln y = 1 + \tan^{-1} x$. Show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$.

Find the first three terms of the Maclaurin's series for y.
Write down the equation of the normal at the point (0,e) on the curve $\ln y = 1 + \tan^{-1} x$.

b. Given that $\ln y = \sin^{-1} x$, show that $\sqrt{1-x^2}\frac{dy}{dx} = y$. By repeated differentiation of this result or otherwise, find the Maclaurin's expansion for y up to and including the term in x^3. Deduce the approximate value of $e^{\frac{\pi}{6}}$.

*c. Find the general solution to the differential equation:

$$\left(\frac{dy}{dx}\right)^2 - 2y(\sin x)\frac{dy}{dx} + y^2 = (y\cos x)^2$$

*d. . Determine the Maclaurin's expansion for $\sec x - \tan x$, up to and including the term in x^3. Show that, to this degree of approximation, $\sec x - \tan x$ can be expressed as $a + b\ln(1+x)$ where a and b are constants to be determined.

*e. (i) Given that $y = \tan x$, show that $\frac{d^2y}{dx^2} = 2y\left(\frac{dy}{dx}\right)$. Hence, find the Maclaurin's series expansion for y, up to and including the term in x^3.

(ii) Using the standard series expansion for ln (1+x) and the Maclaurin's series for y, find the series expansion for ln (1+tanx) in ascending powers of x up to and including the term in x^3.

(iii) Hence show that the first 3 non-zero terms in the expansion of $\frac{sec^2 2x}{1+\tan 2x}$ are $1-2x+8x^2$.

*f. Two variables x and y are connected by the differential equation $\frac{dy}{dx} = \frac{1+x+y}{1-x-y}$, use the substitution u= x+y to solve the differential equation. Deduce that $(x+y)^2 + 2(x-y) = A$, where A is a constant.

SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

c. $\left(\frac{dy}{dx}\right)^2 - 2y(\sin x)\frac{dy}{dx} + y^2 = (y\cos x)^2$

$$\left(\frac{dy}{dx}\right)^2 - 2y(\sin x)\frac{dy}{dx} = y^2(\cos^2 x - 1)$$

$$(\frac{dy}{dx})^2 - 2y(\sin x)\frac{dy}{dx} = -y^2(\sin^2 x)$$

$$(\frac{dy}{dx} - y\sin x)^2 = 0$$

$$\therefore \frac{dy}{dx} = y\sin x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x$$

$$\int \frac{1}{y}dy = \int \sin x dx$$

$$\ln y = -\cos x + c \Rightarrow y = e^{-\cos x + C} = Ae^{-\cos x} \text{ (shown)}$$

d. . Let $f(x) = y = \sec x - \tan x$, then

$$f'(x) = \frac{dy}{dx} = \sec x \tan x - \sec^2 x = \sec x(\tan x - \sec x) = -y\sec x$$

$$f''(x) = \frac{d^2y}{dx^2} = -(\frac{dy}{dx}\sec x + y\sec x \tan x) = (-\sec x)(\frac{dy}{dx} + y\tan x)$$

$$f'''(x) = \frac{d^3y}{dx^3} = (-\sec x)(\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + y\sec^2 x) + (-\sec x \tan x)(\frac{dy}{dx} + y\tan x)$$

$$f(0) = 1, f"(0) = -1, f''(0) = 1, f'''(0) = -2$$

$$\therefore \quad f(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} = 1 - [x - \frac{x^2}{2} + \frac{x^3}{3}] = 1 - \ln(1+x)$$

By comparison, a=1, b= -1 (shown)

e. (i) $y = tanx$

Differentiating both sides wrt x gives

$$\frac{dy}{dx} = sec^2 x = 1 + \tan^2 x = 1 + y^2$$

Differentiating both sides again wrt x gives

$$\frac{d^2y}{dx^2} = 2\text{y}(\frac{dy}{dx}) \text{ (shown)}$$

Differentiating both sides a third time wrt x gives

$$\frac{d^3y}{dx^3} = 2(\frac{dy}{dx})^2 + 2\text{y}(\frac{d^2y}{dx^2})$$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 2$$

Hence, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^{(n)}(0) + ...$

$= x + \frac{2x^3}{3!} = x + \frac{x^3}{3}$ (shown)

(ii) Series expansion for $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$

Hence, $\ln(1+tanx) \approx \ln[1+(x+\frac{x^3}{3})]$

$= (x+\frac{x^3}{3}) - (\frac{1}{2})(x+\frac{x^3}{3})^2 + (\frac{1}{3})(x+\frac{x^3}{3})^3$

Collecting all terms up to and including x^3 gives

$\ln(1+tanx) \approx x + \frac{x^3}{3} - \frac{x^2}{2} + (\frac{1}{3})(x^3) = x - \frac{x^2}{2} + \frac{2x^3}{3}$ (shown)

(iii) $\ln(1+tan2x) = 2x - \frac{(2x)^2}{2} + \frac{2(2x)^3}{3} = 2x - 2x^2 + \frac{16x^3}{3}$

$\frac{d}{dx}[\ln(1+tan2x)] = \frac{d}{dx}[2x - 2x^2 + \frac{16x^3}{3}]$

$\frac{2\sec^2 2x}{1+\tan 2x} = 2 - 4x + 16x^2$

Therefore, $\frac{\sec^2 2x}{1+\tan 2x} = 1 - 2x + 8x^2$ (shown)

f. $u = x + y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$

Substituting into $\frac{dy}{dx} = \frac{1+x+y}{1-x-y}$,

$\frac{du}{dx} = 1 + \frac{1+u}{1-u} = \frac{2}{1-u}$

$\int (1-u)du = 2\int dx$

$u - \frac{u^2}{2} = 2x + C$

$(x+y) - \frac{(x+y)^2}{2} = 2x + C$

$2(x+y) - (x+y)^2 = 4x + B$

$2(y-x) - (x+y)^2 = B$

$\therefore (x+y)^2 + 2(x-y) = A$ (shown)

6. Functions

-ability to prove existence/ non existence of inverse function via
(i) horizontal line test
(ii) proof by contradiction (eg for $f(x)=x^2$, $x \in R$, $f(1)=f(-1)=1 \Rightarrow$ **2 values of x map to the same image** y ie no inverse function exists.)
(Sidenote: a popular feature of questions is to demand students to show that a certain function is **strictly increasing/decreasing** for a certain set of x values- this can be done by finding $f'(x)$ and proving that $f'(x) \geq 0$ when strictly increasing and $f'(x) \leq 0$ when strictly decreasing)

-ability to **derive expressions for inverse functions** and recognise that $f^{-1}(x)$ can be obtained graphically via reflecting $f(x)$ in the line $y=x$; also recognise that $R_{f^{-1}} = D_f$, $D_{f^{-1}} = R_f$

-ability to solve for $f(x)=f^{-1}(x)$ (Note that this is achieved easily by simply solving $f(x)=x$)

-ability to discern if c**omposite function** eg $gf(x)$ exists, and to find the rule and range of the resultant composite function

-ability to **resize domains** of certain single standing functions such that the inverse or composite versions exist.(eg $f(x)=x^2$, $x \in R$, having its domain resized to $x \geq 0$ allows for its inverse to exist).

PREDICTED QUESTION STRUCTURES :

*a. The functions f and g are defined as follows.

$$f : x \rightarrow x^2 - 4, \quad x \in \Re$$
$$g : x \rightarrow \sqrt{x-5}, \quad x \geq 5$$

(i) Explain briefly why the inverse function f^{-1} does not exist.

(ii) State the value of A such that $\{x \in \Re : x \geq A\}$ for the inverse function to exist.

(iii) Give a reason why the composite function gf does not exist. Write down the largest domain of f so that the composite function gf exists. With this restricted domain of f, write down the rule and domain of gf.

*b. The functions f and g are defined by

$$f : x \rightarrow \cos x, \quad -\pi \leq x \leq \pi ,$$

$g: x \to 2x^2 + x, \quad x \geq -1.$

(i) Show that the function gf exists.

(ii) Find the function gf in using the expressions above ands state its range.

(iii) Explain why fg does not exist. Find the largest domain of g such that fg exists.

*c. It is given that

$$f(x) = \begin{cases} 7 - x^2 & \text{for } 0 < x \leq 2 \\ 2x - 1 & \text{for } 2 < x \leq 4 \end{cases}$$

and that $f(x) = f(x+4)$ for all real values of x.

(i) Evaluate $f(27) + f(45)$.
(ii) Sketch the graph of $f(x)$ for $-7 \leq x \leq 10$.
(iii) Find $\int_{-4}^{3} f(x)dx$.

SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX :

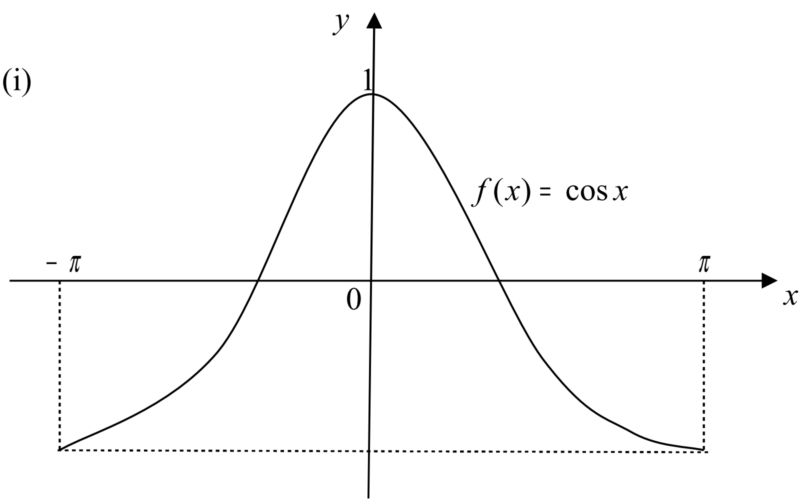
a. (i) $f(2) = f(-2) = 0$, hence $f(x)$ is not 1-1 and $f^{-1}(x)$ does not exist. (shown)

(ii) A=0 (shown)

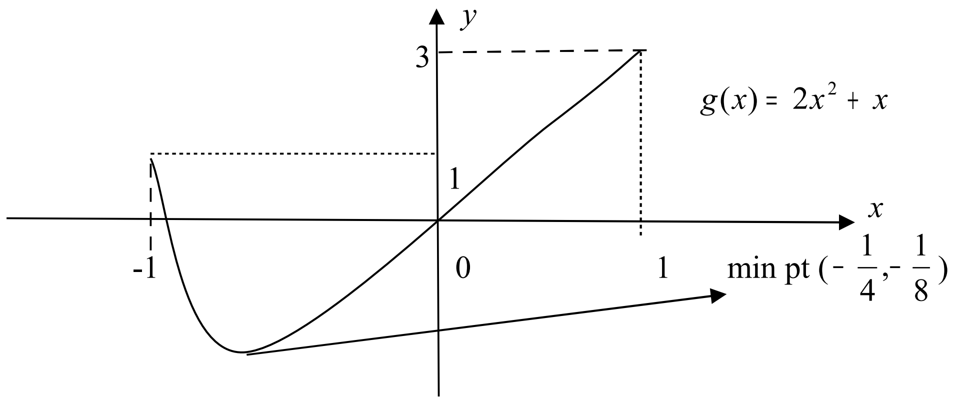
(iii) r_f : [-4, ∞) r_g :[0, ∞)

Since $r_f \not\subset d_g$, hence $gf(x)$ does not exist. (shown)
$r_f \cap d_g = x \geq 5 \Rightarrow$ largest domain of f is $x \geq 3$ or $x \leq -3$ (shown)
$gf: x \to \sqrt{x^2 - 9}$ (shown)

b (i)



-1



$r_f : [-1\ 1]$ $\quad r_g : [-\frac{1}{8}, \infty)$

Since $r_f \subseteq d_g$, $\therefore$ gf exists. (shown)

(ii) $gf(x) = 2\cos^2 x + \cos x$ (shown)

$r_f \cap d_g = [-1, 1]$ Using this as the new input domain for $g(x)$,

new corresponding range = range of $gf(x) = [-\frac{1}{8}, 3]$ (shown)

(iii) $r_g \not\subset d_f$, hence fg does not exist. (shown)

Largest possible range of $g(x)$ allowed based on domain of $f(x) = [-\frac{1}{8}, \pi]$.

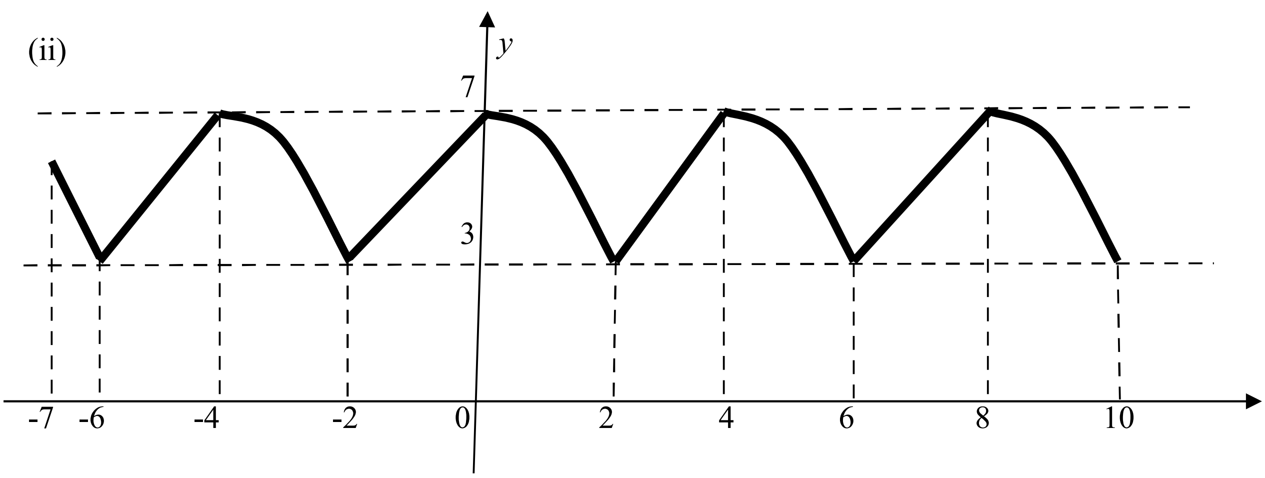
Therefore, the largest possible domain of $g(x)$ based on the above stipulated domain is [-1, 1.03] (shown)

c. (i) $f(27) = f(23) = f(19) = ... = f(3) = 2(3) - 1 = 5$

$f(45) = f(41) = f(37) = ...f(1) = 7 - 1^2 = 6$

$\therefore$ $f(27) + f(45) = 5 + 6 = 11$ (shown)

(ii)



(iii) $\int_{-4}^{3} f(x)dx = 2\left[\int_{0}^{2} 7 - x^2 dx + \frac{1}{2}(2)(3+7)\right] - \int_{3}^{4} 2x - 1 dx$

$$= 2\left\{\left[7x - \frac{x^3}{3}\right]_0^2 + 10\right\} - \left[x^2 - x\right]_3^4$$

$$= 42\frac{2}{3} - 6 = 36\frac{2}{3} \text{ (shown)}$$