Vectors Summary

1. Scalar product (dot product):

$$a \bullet b = |a||b|\cos\theta$$

Laws of dot product:

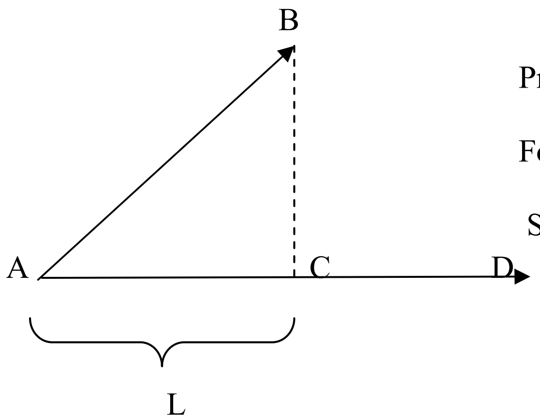
(i) $a \bullet b = b \bullet a$ (ii) $a \bullet (b+c) = a \bullet b + a \bullet c = b \bullet a + c \bullet a$

(iii) $a \bullet a = |a|^2$ (angle between two identical vectors is 0 degrees)

(iv) $a \bullet b = 0 \Rightarrow a$ and b are perpendicular

Applications:

(i) Projection vector:



Length of projection L= $|\overrightarrow{AB} \bullet \overrightarrow{\hat{AD}}|$

Projection vector $\overrightarrow{AC} = (\overrightarrow{AB} \bullet \overrightarrow{\hat{AD}})\ \overrightarrow{\hat{AD}}$

Foot of perpendicular = $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

Shortest distance from B to line

$= |BC|^2 = |AB|^2 - L^2$

[OR = $|\overrightarrow{AB} \times \overrightarrow{\hat{AD}}|$]

(ii) Acute angle between two lines:

$$\theta = \cos^{-1}\left(\frac{|m_1 \bullet m_2|}{|m_1||m_2|}\right)$$

where m_1 and m_2 are the direction vectors of the the two lines.

(iii) Acute angle between two planes:

$$\theta = \cos^{-1}\left(\frac{|n_1 \bullet n_2|}{|n_1||n_2|}\right)$$

where n_1 and n_2 are the individual normals to the two planes respectively.

(iv) Acute angle between a line and a plane:

$$\theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{|m \bullet n|}{|m||n|}\right)$$

where m and n are the direction vector of the line and normal to the plane respectively, and θ is the angle between the line and plane in question.

2. Cross product (vector product):

$$\boxed{a \times b = [|a||b|\sin\theta]\hat{n}}$$ where n is a vector that is perpendicular to both a and b.

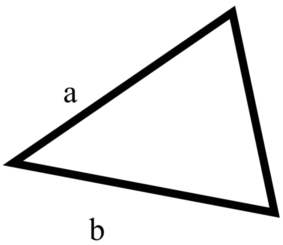
<u>Laws of cross product:</u>

(i) $a \times b = -(b \times a)$ (ii) $a \times (b + c) = a \times b + a \times c = -(b \times a) - (c \times a)$

(iii) $a \times a = \tilde{0}$

<u>Applications:</u>

(i)



Area of triangle$= \frac{1}{2}|a \times b|$

(ii) If four points A, B, C and D are **coplanar**, then $|\overrightarrow{AB} \times \overrightarrow{AC}| \bullet \overrightarrow{AD} = 0$

3. Equation of lines:

<u>Representations:</u>

(i) $r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ (parametric form) OR $r = a + \lambda m$ (condensed form)

(ii) $\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$ (cartesian form)

4. Equations of planes:

<u>Representations:</u>

(i) $r = a + \lambda m_1 + \mu m_2$ (parametric form)

(ii) $r \bullet n = a \bullet n$ (scalar product form)

(where $n = m_1 \times m_2$, a is the **position vector of a point lying on the plane.)**

(iii) $ax + by + cz = k$ (Cartesian form)

(where a, b and c are the components of the normal vector to the plane)

5. Skew lines:

Two lines with equations $r = a + \lambda m_1$ and $r = b + \mu m_2$ are said to be skew lines if they **DO NOT** intersect at a common point and m_1 is **NOT PARALLEL** to m_2.

6. Determining if line resides in plane:

A line with equation $r = a + \lambda m$ is said to reside in the plane $r \bullet n = k$ if

(i) $m \bullet n = 0$ (ii) $a \bullet n = k$

7. Shortest distance from plane to origin:

For a plane with equation $r \bullet n = k$, the shortest distance from the plane to the origin is given by $\left|\frac{k}{\|n\|}\right|$.

8. Distance between 2 planes:

For 2 planes with equations $r \bullet n = k_1$ and $r \bullet n = k_2$, where $|k_1| < |k_2|$, the shortest distance between them is given by:

(i) $\left|\frac{k_1}{\|n\|}\right| + \left|\frac{k_2}{\|n\|}\right|$ if k_1 and k_2 are of **different signs**

(ii) $\left|\frac{k_2}{\|n\|}\right| - \left|\frac{k_1}{\|n\|}\right|$ if k_1 and k_2 are of the **same signs**

9. Finding intersection between various constructs:

<u>(i) Intersection between 2 lines:</u>

For 2 lines with equations $r = a + \lambda m_1$ and $r = b + \mu m_2$, equate them to each other in column vector form such that $a + \lambda m_1 = b + \mu m_2$. Solve for the values of λ and μ before substituting back into either of the two line equations to derive the common point of intersection.

<u>(ii) Intersection between line and plane:</u>

For a line with equation $r = a + \lambda m$ and a plane with equation $r \bullet n = k$, substitute the line equation within that of the plane equation such that $(a + \lambda m) \bullet n = k$. Solve for the value of λ and subsequently derive the common point of intersection through substitution of λ into the line equation.

<u>(iii) intersection between 2 planes:</u>

A. If one plane is presented in **scalar product form** and the other in **parametric form,**

Example: $r \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 6$ ------------------------(1)

$$r = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{-----------(2)}$$

$$\Rightarrow \begin{pmatrix} 1 + 3\lambda + \mu \\ 3\lambda + \mu \\ 1 + \lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 6$$

$$1+3\lambda+\mu+9\lambda+3\mu+1+\lambda=6$$
$$12\lambda+4\mu=4$$
$$3\lambda+\mu=1\rightarrow\mu=1-3\lambda$$

Substituting this back into (2),

Equation of line of intersection is

$$r=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\3\\1\end{pmatrix}+(1-3\lambda)\begin{pmatrix}1\\1\\0\end{pmatrix}$$
$$=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\text{(shown)}$$

B. If **both** planes are presented in **Cartesian form**:

Example: $x+y+z=9$ ----------(1)

$-x-y+z=1$ ----------(2)

(1)+(2): $2z=10\Rightarrow z=5$

Let $y=t$ and substituting this together with $z=5$ into (1),

We have $x=4-t$

Equation of line of intersection is

$$r=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4-t\\t\\5\end{pmatrix}=\begin{pmatrix}4\\0\\5\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}\quad\text{(shown)}$$

C. If **both** planes are presented in **scalar product forms** or **in parametric forms** or one is presented in **scalar product form** and the **other in parametric form,** convert the plane equations such that their configurations matches that of either case A or B, and solve accordingly.

D. If a common point A with position vector a is known to reside on both planes, and the two planes have normal vectors n_1 and n_2, then the common line of intersection is simply given by $r=a+\lambda(n_1\times n_2)$.

<u>(iv) Intersection between 3 planes:</u>

Extract the components of the separate plane equations to form the **augmented matrix:**

$$r \bullet \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = d_1 , \quad r \bullet \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = d_2 \quad r \bullet \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = d_3$$

$$\downarrow$$

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

After reducing the augmented matrix to its row reduced equivalent **using the RREF function** of the graphic calculator, 3 possible scenarios arise:

A. The planes intersect at one point, ie there is a unique solution to the matrix.

Example:

$$\begin{pmatrix} 2 & -1 & 1 & 4 \\ 1 & 2 & -2 & -3 \\ -4 & 2 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$1(x) + 0(y) + 0(z) = 1$, therefore $x = 1$,
$0(x) + 1(y) + 0(z) = 2$, therefore $y = 2$,
$0(x) + 0(y) + 1(z) = 4$, therefore $z = 4$

Hence, the 3 planes **intersect** at the point (1,2,4).

B. The three planes do not intersect at all.

Example:

$$\begin{pmatrix} 1 & 2 & -2 & -2 \\ -1 & 2 & -1 & 5 \\ 1 & -6 & 4 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.75 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the third row in the reduced form matrix, 0=1, giving rise to a contradiction, hence there is no common point to the 3 planes, ie they **DO NOT intersect**.

C. The three planes intersect at a line.

Example:

$$\begin{pmatrix} 1 & 2 & -2 & -2 \\ -1 & 2 & -1 & 5 \\ 1 & -6 & 4 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -0.5 & -3.5 \\ 0 & 1 & -0.75 & 0.75 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the reduced row matrix, we have

$$x - \frac{1}{2}z = -\frac{7}{2} \Rightarrow x = -\frac{7}{2} + \frac{1}{2}z,$$

$$y - \frac{3}{4}z = \frac{3}{4} \Rightarrow y = \frac{3}{4} + \frac{3}{4}z$$

Let $z = \lambda$, then $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ \frac{3}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ \frac{3}{4} \\ 0 \end{pmatrix} + \frac{1}{4}\lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Therefore, the three planes **intersect** at the **line r**$= \begin{pmatrix} -\frac{7}{2} \\ \frac{3}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, where $t = \frac{\lambda}{4} \in \Re$