Series/Sequences and Mathematical Induction Summary

1. Summation

(a) Anatomy of the sigma notation:

$$\sum_{r=a}^{n} 5r+2^r = \underbrace{\left[5(a)+2^a\right]}_{r=a}+\underbrace{\left[5(a+1)+2^{a+1}\right]}_{r=a+1}+\underbrace{\left[5(a+2)+2^{a+2}\right]}_{r=a+2}+\ldots\ldots..+\underbrace{\left[5(n)+2^n\right]}_{r=n}$$

r is the **variable**; it **changes value** from a (for the first term of the series) to n (for the last term of the series). In this particular context both a and n are **fixed integer constants**.

Total number of terms $= n-a+1$

Note: r need not always necessarily be assigned as the variable, so be mindful of the representations given in the question and interpret the series structure expansion correctly.

Consider the immediate 3 examples given below where shuffling in terms of naming the components have been made:

(i) $$\sum_{a=n}^{r} 5a+2^a = \left[5(n)+2^n\right]+\left[5(n+1)+2^{n+1}\right]+\left[5(n+2)+2^{n+2}\right]+\ldots\ldots..+\left[5(r)+2^r\right]$$

(ii) $$\sum_{a=n}^{r} 5n+2^n = \underbrace{\left[5(n)+2^n\right]+\left[5(n)+2^n\right]+\left[5(n)+2^n\right]+\ldots\ldots..+\left[5(n)+2^n\right]}_{r-n+1 \quad \text{terms}}$$

(iii) $$\sum_{n=a}^{r} 5n+2^n = \left[5(a)+2^a\right]+\left[5(a+1)+2^{a+1}\right]+\left[5(a+2)+2^{a+2}\right]+\ldots\ldots..+\left[5(r)+2^r\right]$$

(b) Formulas of popular summation series:

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1) \qquad \sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1) \qquad \sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2 \quad \left[=\left(\sum_{r=1}^{n} r\right)^2\right]$$

$$\sum_{r=1}^{n} \ln r = \ln 1+\ln 2+\ln 3+\ldots\ldots..+\ln n = \ln(1\times 2\times 3\times\ldots\ldots\ldots..\times n) = \ln(n!)$$

Note: $\sum_{r=2}^{n} \ln r$ also $= \ln(n!)$ since $\ln 1 = 0$

$$\sum_{r=1}^{n} a^r = \frac{a(1-a^n)}{1-a} = \frac{a(a^n-1)}{a-1}$$

Applying these formulas in a strictly direct manner **requires the lower bound to start from the value of 1**. (This is with the exception of the natural logarithm series) Hence, adjustments must be made when this requirement is not met.

Example: $\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 = \frac{2n}{6}(2n+1)(4n+1) - \frac{n-1}{6}[(n-1)+1][2(n-1)+1]$

$$= \frac{n}{3}(2n+1)(4n+1) - \frac{n-1}{6}(n)(2n-1)$$

The condensed sigma notation may also be equivalent to certain **Arithmetic Progressions** and **Geometric Progressions**.

AP: $\sum_{r=a}^{b} f(r)$ where $f(r)$ must be a linear function in r.

Example: $\sum_{r=5}^{n} 3r-1 = 14+17+20+23+ \ldots\ldots\ldots\ldots + (3n-1)$ is an AP with first term 14,

common difference of 3 and a total of $n-5+1 = n-4$ terms.

Hence, $\sum_{r=5}^{n} 3r-1$ can be simplified to give $\frac{n-4}{2}[14+(3n-1)] = \frac{n-4}{2}(13+3n)$

In such an instance, the lower bound **DOES NOT** need to start from 1.

GP: $\sum_{r=a}^{b} k^{g(r)}$ where k is a real number constant and $g(r)$ must be a linear function in r.

$\sum_{r=3}^{2n} 2^{2r+1} = 2^7 + 2^9 + 2^{11} + \ldots\ldots\ldots\ldots\ldots\ldots + 2^{4n+1}$ is a GP with first term 2^7, common ratio of

$2^2 = 4$ and a total of $2n-3+1 = 2n-2$ terms.

Hence, $\sum_{r=3}^{2n} 2^{2r+1}$ can be simplified to give $\frac{2^7\left(4^{2n-2}-1\right)}{4-1} = \frac{128}{3}\left(4^{2n-2}-1\right)$

In such an instance, the lower bound **DOES NOT** need to start from 1.

(c) Basic operations:

(i) $\sum_{r=a}^{b} f(r) \pm g(r) = \sum_{r=a}^{b} f(r) \pm \sum_{r=a}^{b} g(r)$

(ii) $\sum_{r=a}^{b} kf(r) = k\sum_{a}^{b} f(r)$ where k is a real constant.

(iii) $\sum_{r=a}^{b} f(r) + \sum_{r=b+1}^{2b} f(r) = \sum_{r=a}^{2b} f(r)$ or $\sum_{r=a}^{b} f(r) = \sum_{r=a}^{2b} f(r) - \sum_{r=b+1}^{2b} f(r)$

(iv) $\sum_{r=a}^{b} h(k) = (b-a+1)\left[h(k)\right]$ (Note that every term of the series is simply $h(k)$)

(v) If $f(a) = 0$ and $f(a+1)$, $f(a+2)$, $f(a+3)$, ...$f(b)$ are all non-zero terms,

Then $\sum_{r=a}^{b} f(r) = \sum_{r=a+1}^{b} f(r)$

2. Method of Differences

When partial fractions are typically involved along with the employment of the sigma notation, there is a very high likelihood MOD must be a main part of the solving strategy.

A mass cancellation shall be effected and the number of (surviving) terms normally collected would be an **even** quantity; of course this would depend on the number of linear factors constituting the denominator of the original block to be broken up by the partial fractions method.

Example: Express $\frac{r}{(r+1)(r+2)(r+3)}$ in partial fractions, and hence show

that $\sum_{r=1}^{n} \frac{r}{(r+1)(r+2)(r+3)} = \frac{1}{4} + \frac{1}{2(n+2)} - \frac{3}{2(n+3)}$.

SOLUTIONS :

$$\frac{r}{(r+1)(r+2)(r+3)} = \frac{\left(-\frac{1}{2}\right)}{r+1} + \frac{2}{r+2} + \frac{\left(-\frac{3}{2}\right)}{r+3}$$

$$= \frac{1}{2}\left[-\frac{1}{r+1} + \frac{4}{r+2} - \frac{3}{r+3}\right] \text{ (shown)}$$

$$\sum_{r=1}^{n}\frac{r}{(r+1)(r+2)(r+3)}=\frac{1}{2}\left\{\begin{array}{l}
-\frac{1}{2}+\frac{4}{3}-\frac{3}{4}\\[4pt]
-\frac{1}{3}+\frac{4}{4}-\frac{3}{5}\\[4pt]
-\frac{1}{4}+\frac{4}{5}-\frac{3}{6}\\[4pt]
-\frac{1}{5}+\frac{4}{6}-\frac{3}{7}\\[4pt]
\qquad\vdots\\[4pt]
\qquad\qquad -\frac{3}{(n-1)+3}\\[4pt]
-\frac{1}{n+1}+\frac{4}{n+2}-\frac{3}{n+3}
\end{array}\right.$$

$$=\frac{1}{2}\left(-\frac{1}{2}+\frac{4}{3}-\frac{1}{3}+\frac{4}{n+2}-\frac{3}{n+2}-\frac{3}{n+3}\right)\quad=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{n+2}-\frac{3}{n+3}\right)$$

$$=\frac{1}{4}+\frac{1}{2(n+2)}-\frac{3}{2(n+3)}\quad\text{(shown)}$$

3. Mathematical Induction

(a) Basic full structure outline:

Let P_n be the proposition that……………………

For P_1: $LHS = ...$ $RHS = ...$

Since $LHS = RHS$, P_1 is true.

(Note: at times we may start from P_2 or P_0)

Assume P_k is true for some $k \in Z^+$, ie ……………………

Looking at P_{k+1} :…………………………………….. (THIS IS THE MAIN BODY, AND THE ABOVE ASSUMPTION **MUST ALWAYS BE SUBSTITUTED** INTO THIS PART IN ONE WAY OR ANOTHER)

Conclusion: P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, by **Mathematical Induction**, P_n is true for all $n \in Z^+$. (Note that if we start from P_2, then we have to revise the truth criteria to $n \geq 2$; if we start from P_0, we have to revise this to $n \geq 0$)

<u>(b) Typical types of MI question structures:</u>

(i) Testing the validity of a formula describing the summation of a series:

Example: Use induction to prove that $\sum_{r=2}^{n}(r^2+r+1)r! = (n+1)^2 n! - 4$.

<u>SOLUTION FOR THE MAIN BODY:</u>

Assume P_k is true, ie $\sum_{r=2}^{k}(r^2+r+1)r! = (k+1)^2 k! - 4$

For P_{k+1} : $\sum_{r=2}^{k+1}(r^2+r+1)r! = (k+1)^2 k! - 4 + \left[(k+1)^2+(k+1)+1\right](k+1)!$

$$= (k+1)\left[(k+1)k!\right] + \left[(k+1)^2+(k+1)+1\right](k+1)! - 4$$

$$= (k+1)(k+1)! + \left[(k+1)^2+(k+1)+1\right](k+1)! - 4$$

$$= (k+1)!\left[k+1+k^2+3k+3\right] - 4$$

$$= (k+1)!\left[k^2+4k+4\right] - 4$$

$$= (k+1)!(k+2)^2 - 4 = (k+1)!\left[(k+1)+1\right]^2 - 4 \text{ (shown)}$$

(ii) Testing the validity of a formula **describing the generic term** within a series:

Example: A sequence $u_0, u_1, u_2, ...$ is defined by $u_0 = -3$ and $u_{n+1} = 2u_n + 3^n + 5n$ for $n \geq 0$. Prove by mathematical induction that for all $n \geq 0$, $u_n = 2^n + 3^n - 5n - 5$.

SOLUTION FOR THE MAIN BODY:

Assume P_k is true, ie $u_k = 2^k + 3^k - 5k - 5$

For P_{k+1}: $u_{k+1} = 2u_k + 3^k + 5k$

$$= 2\left(2^k + 3^k - 5k - 5\right) + 3^k + 5k$$

$$= 2^{k+1} + 2\left(3^k\right) - 10k - 10 + 3^k + 5k$$

$$= 2^{k+1} + (2+1)\left(3^k\right) - 5k - 10$$

$$= 2^{k+1} + (3)\left(3^k\right) - \left(5k - 5\right) - 5$$

$$= 2^{k+1} + 3^{k+1} - 5(k+1) - 5 \quad \text{(shown)}$$

<u>(c) Atypical question structure types:</u>

Unexpected nasty problems could surface, in such cases we must be astute and sufficiently competent to work out the proof.

Example: Using the formula for $\sin(A \pm B)$, prove that

$$\sin\left(r + \frac{1}{2}\right)\theta - \sin\left(r - \frac{1}{2}\right)\theta \equiv 2\cos r\theta \sin\frac{1}{2}\theta.$$

Hence find a formula for $\sum_{r=1}^{n} \cos r\theta$ in terms of $\sin\left(n + \frac{1}{2}\right)\theta$ and $\sin\frac{1}{2}\theta.$

Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

for all positive integers n.

FULL SOLUTIONS :

$$\sin\left(r + \frac{1}{2}\right)\theta - \sin\left(r - \frac{1}{2}\right)\theta = \sin\left(r\theta + \frac{1}{2}\theta\right) - \sin\left(r\theta - \frac{1}{2}\theta\right)$$

$$= \sin r\theta \cos\frac{1}{2}\theta + \cos r\theta \sin\frac{1}{2}\theta - \left(\sin r\theta \cos\frac{1}{2}\theta - \cos r\theta \sin\frac{1}{2}\theta \right)$$

$$= 2\cos r\theta \sin\frac{1}{2}\theta \text{ (shown)}$$

$$\sum_{r=1}^{n} \left[\sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta \right] = \sum_{r=1}^{n} 2\cos r\theta \sin\frac{1}{2}\theta$$

$$\sum_{r=1}^{n} \cos r\theta = \frac{1}{2\sin\frac{1}{2}\theta} \sum_{r=1}^{n} \left[\sin\left(r+\frac{1}{2}\right)\theta - \sin\left(r-\frac{1}{2}\right)\theta \right]$$ (note that $2\sin\frac{1}{2}\theta$ is a constant and can be isolated outside the sigma notation.)

$$= \frac{1}{2\sin\frac{1}{2}\theta} \left\{ \begin{array}{l} \sin\frac{3}{2}\theta - \sin\frac{1}{2}\theta \\ \sin\frac{5}{2}\theta - \sin\frac{3}{2}\theta \\ \sin\frac{7}{2}\theta - \sin\frac{5}{2}\theta \\ \quad \vdots \\ \sin\left(n+\frac{1}{2}\right)\theta - \sin\left(n-\frac{1}{2}\right)\theta \end{array} \right. = \frac{1}{2\sin\frac{1}{2}\theta} \left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta \right] \text{ (shown)}$$

Let P_n be the hypothesis that $\sum_{r=1}^{n} \sin r\theta = \dfrac{\cos\frac{1}{2}\theta - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$, $n \in \mathbb{Z}^{+}$.

For P_1 : $LHS = \sin\theta$; $RHS = \dfrac{\cos\frac{1}{2}\theta - \cos\left(\frac{3}{2}\right)\theta}{2\sin\frac{1}{2}\theta} = \dfrac{\cos\left(\frac{3}{2}\right)\theta - \cos\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta}$

$$= \frac{-2\sin\theta\sin\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta} = \sin\theta$$

[$\because$ $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ is used above]

$LHS = RHS$, $\therefore$ P_1 is true.

Assume P_k is true form some $k \in \mathbb{Z}^+$, ie $\displaystyle\sum_{r=1}^{k} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$

For P_{k+1} : $\displaystyle\sum_{r=1}^{k+1} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta} + \sin(k+1)\theta$

$$= \frac{\cos\frac{1}{2}\theta - \cos\left(k+\frac{1}{2}\right)\theta + 2\sin(k+1)\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} \text{-------- (1)}$$

Using a variation of the formula $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

ie $2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right) = \cos A - \cos B$,

where $A = \left(k+\frac{1}{2}\right)\theta$ and $B = \left(k+\frac{3}{2}\right)\theta$.

(1) becomes $\dfrac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{1}{2}\right)\theta+\cos\left(k+\frac{1}{2}\right)\theta-\cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$

$$=\frac{\cos\frac{1}{2}\theta-\cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{1}{2}\theta}=\frac{\cos\frac{1}{2}\theta-\cos\left[(k+1)+\frac{1}{2}\right]\theta}{2\sin\frac{1}{2}\theta}$$

P_k is true $\Rightarrow P_{k+1}$ is true; since P_1 is true, therefore by mathematical induction,

$\displaystyle\sum_{r=1}^{n}\sin r\theta=\frac{\cos\frac{1}{2}\theta-\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$ is true for all positive integers n. (shown)