

Integration Techniques Summary

1. List of basic formulas:

Function	Integrated Result
ax^n	$\frac{ax^{n+1}}{n+1}$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{(a)(n+1)}$
$[f'(x)][f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1}$
$\frac{[f'(x)]}{a^2 + [f(x)]^2}$	$\frac{1}{a} \tan^{-1} \left[\frac{f(x)}{a} \right]$
$\frac{[f'(x)]}{[f(x)]^2 - a^2}$	$\frac{1}{2a} \ln \left \frac{f(x) - a}{f(x) + a} \right $
$\frac{[f'(x)]}{a^2 - [f(x)]^2}$	$\frac{1}{2a} \ln \left \frac{a + f(x)}{a - f(x)} \right $
$\frac{[f'(x)]}{\sqrt{a^2 - [f(x)]^2}}$	$\sin^{-1} \left[\frac{f(x)}{a} \right]$
$f'(x)a^{f(x)}$	$\frac{1}{\ln a} a^{f(x)}$
$f'(x)e^{f(x)}$	$e^{f(x)}$
$\frac{f'(x)}{f(x)}$	$\ln f(x) $
Function	Integrated Result

$f'(x) \cos[f(x)]$	$\sin[f(x)]$
$f'(x) \sin[f(x)]$	$-\cos[f(x)]$
$f'(x) \sec^2[f(x)]$	$\tan[f(x)]$
$f'(x) \operatorname{cosec}^2[f(x)]$	$-\cot[f(x)]$
$f'(x) \operatorname{cosec}[f(x)] \cot[f(x)]$	$-\operatorname{cosec}[f(x)]$
$\tan x$	$-\ln \cos x $
$\cot x$	$\ln \sin x $
$\sec x$	$\ln \sec x + \tan x $
$\operatorname{cosec} x$	$\ln \operatorname{cosec} x - \cot x $

2. Miscellaneous trigonometric integrals:

a. $\int \sin^n x dx$ and $\int \cos^n x dx$:

If n is **odd**, separate a single $\sin x$ or $\cos x$ from the original function, and use the identity $\sin^2 x + \cos^2 x = 1$.

Example:
$$\int \sin^3 x dx = \int (\sin^2 x)(\sin x) dx = \int (1 - \cos^2 x)(\sin x) dx$$

$$= \int \sin x - \cos^2 x \sin x dx = -\cos x + \frac{\cos^3 x}{3} + C \text{ (shown)}$$

If n is **even**, use the double angle formula of either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$ for conversion.

Example:
$$\int \cos^2 x dx = \int \frac{\cos 2x + 1}{2} dx = \frac{1}{4} \sin 2x + \frac{1}{2} x + C \text{ (shown)}$$

b. $\int \tan^n x dx$:

Separate $\tan^2 x$ from the original function, and use the identity $1 + \tan^2 x = \sec^2 x$.

$$\begin{aligned}
\text{Example: } \int \tan^5 x dx &= \int (\tan^2 x)(\tan^3 x) dx = \int (\sec^2 x - 1)(\tan^3 x) dx \\
&= \int \sec^2 x \tan^3 x - \tan^3 x dx \\
&= \int \sec^2 x \tan^3 x - \tan^2 x(\tan x) dx \\
&= \int \sec^2 x \tan^3 x - (\sec^2 x - 1)(\tan x) dx \\
&= \int \sec^2 x \tan^3 x - \sec^2 x \tan x + \tan x dx \\
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C \quad (\text{shown})
\end{aligned}$$

c. $\int \sin(mx)\sin(nx)dx$ or $\int \sin(mx)\cos(nx)dx$ or $\int \cos(mx)\cos(nx)dx$:

Use one of the 3 identities below to transform the product to a sum or difference:

(i) $\sin(mx)\cos(nx) = \frac{1}{2}\{\sin[(m+n)x] + \sin[(m-n)x]\}$

(ii) $\sin(mx)\sin(nx) = \frac{1}{2}\{\cos[(m-n)x] - \cos[(m+n)x]\}$

(iii) $\cos(mx)\cos(nx) = \frac{1}{2}\{\cos[(m-n)x] + \cos[(m+n)x]\}$

Example: $\int \cos 3x \cos x dx = \frac{1}{2} \int \cos 4x + \cos 2x dx = \frac{1}{8} \sin 4x + \frac{1}{4} \sin 2x + C$ (shown)

3. Integration via substitution:

Generally, to find an integral by means of a substitution $x = f(u)$,

(i) Differentiate x wrt u to arrive at $\frac{dx}{du} = f'(u) \Rightarrow dx = f'(u)du$

(ii) Subsequently replace all x by $f(u)$ and dx by $f'(u)du$

(iii) Perform the integration and remember to convert the result back to x for an indefinite integral (**not needed** for definite integrals)

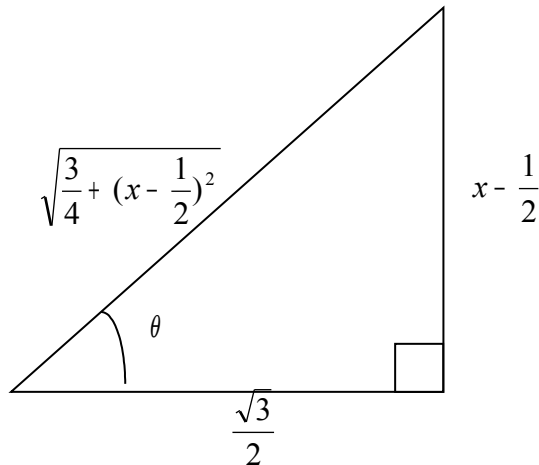
Example: Evaluate $\int \frac{dx}{\sqrt{x^2 - x + 1}}$ by considering the substitution $x - \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$.

$$\int \frac{dx}{\sqrt{x^2 - x + 1}} = \int \frac{1}{\sqrt{(x - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}} dx$$

Using the substitution $x - \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$, the integral becomes

$$\int \frac{1}{\sqrt{\frac{3}{4}\tan^2\theta + \frac{3}{4}}} \left(\frac{\sqrt{3}}{2}\sec\theta\right) d\theta = \int \frac{1}{\frac{\sqrt{3}}{2}\sec\theta} \left(\frac{\sqrt{3}}{2}\sec\theta\right) d\theta = \int (\sec\theta) d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{2}{\sqrt{3}}\left[\sqrt{\frac{3}{4} + \left(x - \frac{1}{2}\right)^2}\right] + \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)\right| + C \text{ (shown)}$$



4. Integration by parts:

$\int uv' dx = uv - \int u'v dx$ where u is a function which can be differentiated and v is a function that can be easily reduced via integration. Note that integration by parts is only feasible if out of the product of two functions, **at least one is directly integrable**.

Example:

$$\begin{aligned} \int x \sin^{-1}(x^2) dx &= \frac{x^2}{2} (\sin^{-1} x^2) - \int \frac{x^2}{2} \left(\frac{1}{\sqrt{1-x^4}}\right) (2x) dx \\ &= \frac{x^2}{2} (\sin^{-1} x^2) - \int \left(\frac{x^3}{\sqrt{1-x^4}}\right) dx \\ &= \frac{x^2}{2} (\sin^{-1} x^2) - \frac{1}{4} \int \left(\frac{4x^3}{\sqrt{1-x^4}}\right) dx = \frac{x^2}{2} (\sin^{-1} x^2) - \frac{1}{4} (2)\sqrt{1-x^4} + C \\ &= \frac{x^2}{2} (\sin^{-1} x^2) - \frac{1}{2} \sqrt{1-x^4} + C \text{ (shown)} \end{aligned}$$

5. Integrals of the form $\int \frac{cx+d}{ax^2+bx+c} dx$

(i) Rewrite $\int \frac{cx+d}{ax^2+bx+c} dx$ as $\int \frac{P(2ax+b)+Q}{ax^2+bx+c} = \int \frac{P(2ax+b)}{ax^2+bx+c} + \frac{Q}{ax^2+bx+c} dx$, where $2ax+b$ is clearly the derivative of ax^2+bx+c .

(ii) $\int \frac{P(2ax+b)}{ax^2+bx+c}$ gives $P \ln|ax^2+bx+c|$

(iii) $\int \frac{Q}{ax^2 + bx + c} dx$ can be easily integrated via completing the square method for the denominator.

Example:
$$\int \frac{x-1}{x^2+x+1} dx = \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx - \frac{3}{2} \int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{2} \ln|x^2+x+1| - \frac{3}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$= \frac{1}{2} \ln|x^2+x+1| - \frac{3}{2} \left(\frac{1}{\frac{\sqrt{3}}{2}}\right) \tan^{-1} \left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2} \ln|x^2+x+1| - \sqrt{3} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}}\right) + C \text{ (shown)}$$

6. Integrals of the form $\int \frac{cx^2 + dx + e}{ax^2 + bx + c} dx$

(i) Rewrite
$$\int \frac{cx^2 + dx + e}{ax^2 + bx + c} dx = \int \frac{P(ax^2 + bx + c) + Q(2ax + b) + R}{ax^2 + bx + c} dx$$

$$= \int P + \frac{Q(2ax + b)}{ax^2 + bx + c} + \frac{R}{ax^2 + bx + c} dx$$

$$= Px + \int \frac{Q(2ax + b)}{ax^2 + bx + c} + \frac{R}{ax^2 + bx + c} dx$$

where $2ax + b$ is clearly the derivative of $ax^2 + bx + c$.

(ii) $\int \frac{Q(2ax + b)}{ax^2 + bx + c}$ gives $Q \ln|ax^2 + bx + c|$

(iii) $\int \frac{R}{ax^2 + bx + c} dx$ can be easily integrated via completing the square method for the denominator.

Example:
$$\int \frac{2x^2 + 4x + 1}{x^2 + x + 1} dx = \int \frac{2(x^2 + x + 1) + (2x + 1) - 2}{x^2 + x + 1} dx$$

$$= 2x + \int \frac{2x + 1}{x^2 + x + 1} - \frac{2}{x^2 + x + 1} dx$$

$$\begin{aligned} &= 2x + \ln|x^2 + x + 1| - 2 \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\ &= 2x + \ln|x^2 + x + 1| - 2 \left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right] + C \\ &= 2x + \ln|x^2 + x + 1| - \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right) + C \text{ (shown)} \end{aligned}$$