<u>Graphing Techniques Summary</u>

A. Fundamental knowledge of the following graphs is **highly essential**:

1. Graphs of the form $y = \dfrac{ax+b}{cx+d}$

Example: $y = \dfrac{x+1}{x-1}$

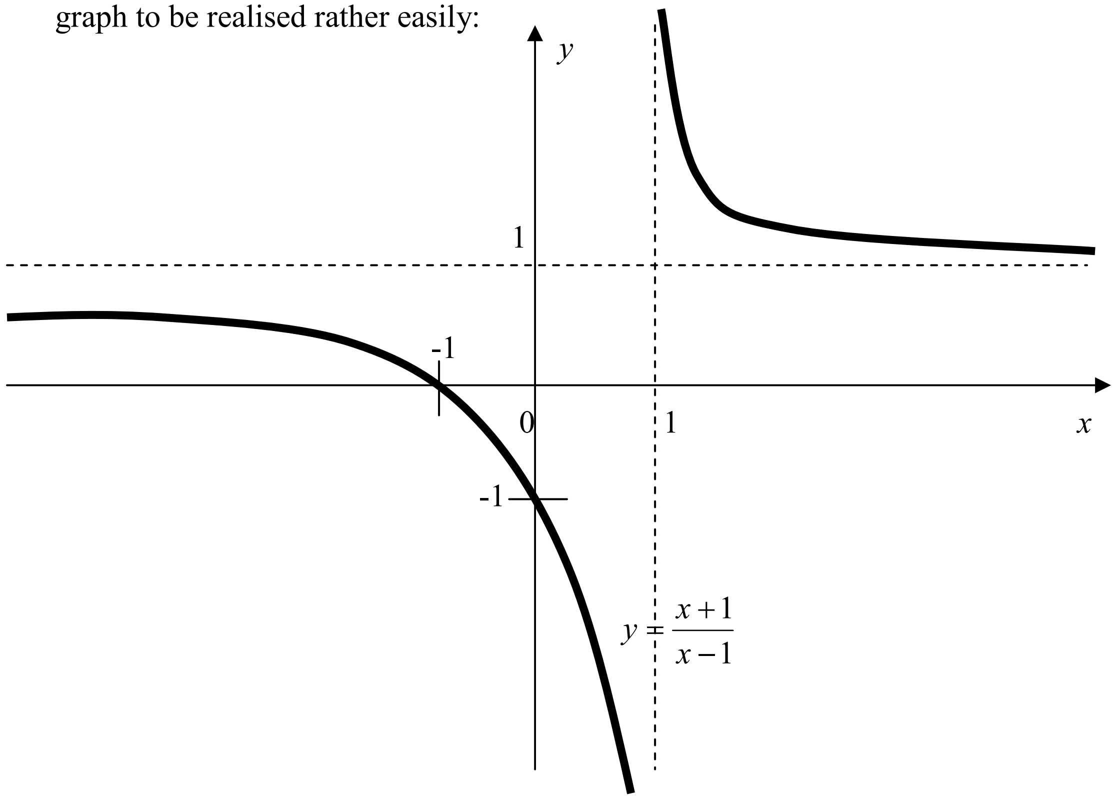
The asymptotes for this equation are $x = 1$ and $y = 1$; this is achieved by having $y \to \pm\infty$ and $x \to \pm\infty$ respectively.

The workings are as follows:

$x - 1 = \dfrac{x+1}{y}$; as $y \to \pm\infty$, $x - 1 \to 0 \Rightarrow x = 1$ is a vertical asymptote

$y = \left(\dfrac{x+1}{x-1}\right)\left(\dfrac{\frac{1}{x}}{\frac{1}{x}}\right) = \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$; as $x \to \pm\infty$, $y \to \dfrac{1+0}{1-0} = 1 \Rightarrow y = 1$ is a horizontal asymptote

Using this information, coupled with the relevant x and y intercepts, allows the graph to be realised rather easily:



2. Graphs of the form $y = \dfrac{ax^2 + bx + c}{ex + f}$

Example: $y = \dfrac{x^2 + 2}{x - 1}$

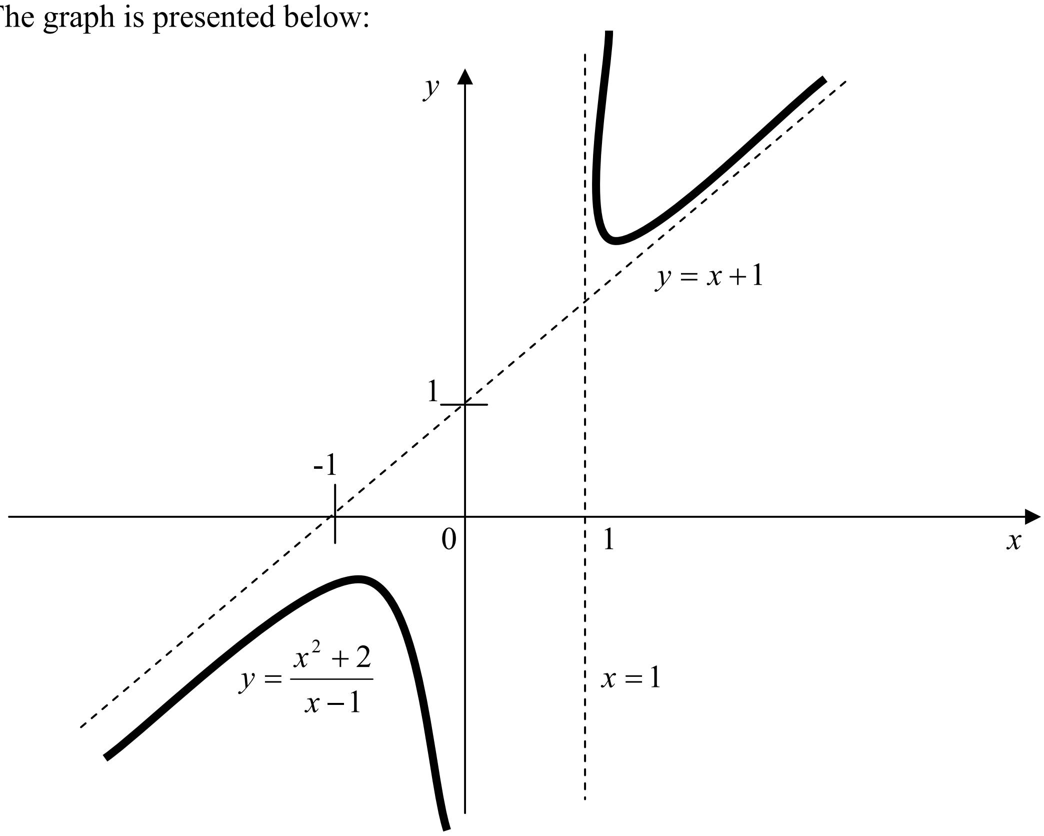
To find the asymptotes:

$x - 1 = \dfrac{x^2 + 2}{y}$; as $y \to \pm\infty$, $x - 1 \to 0 \Rightarrow x = 1$ is a vertical asymptote

$y = \dfrac{x^2 + 2}{x - 1} = \dfrac{x(x-1) + (x-1) + 3}{x - 1} = x + 1 + \dfrac{3}{x - 1}$; as $x \to \pm\infty$,

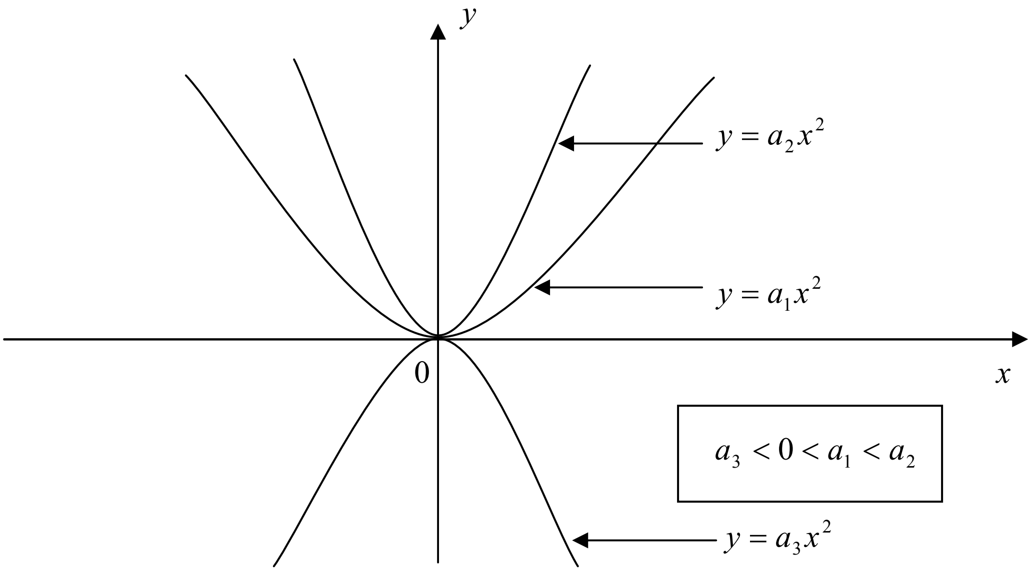
$y \to x + 1 + 0 \Rightarrow y = x + 1$ is an **oblique** asymptote.

The graph is presented below:

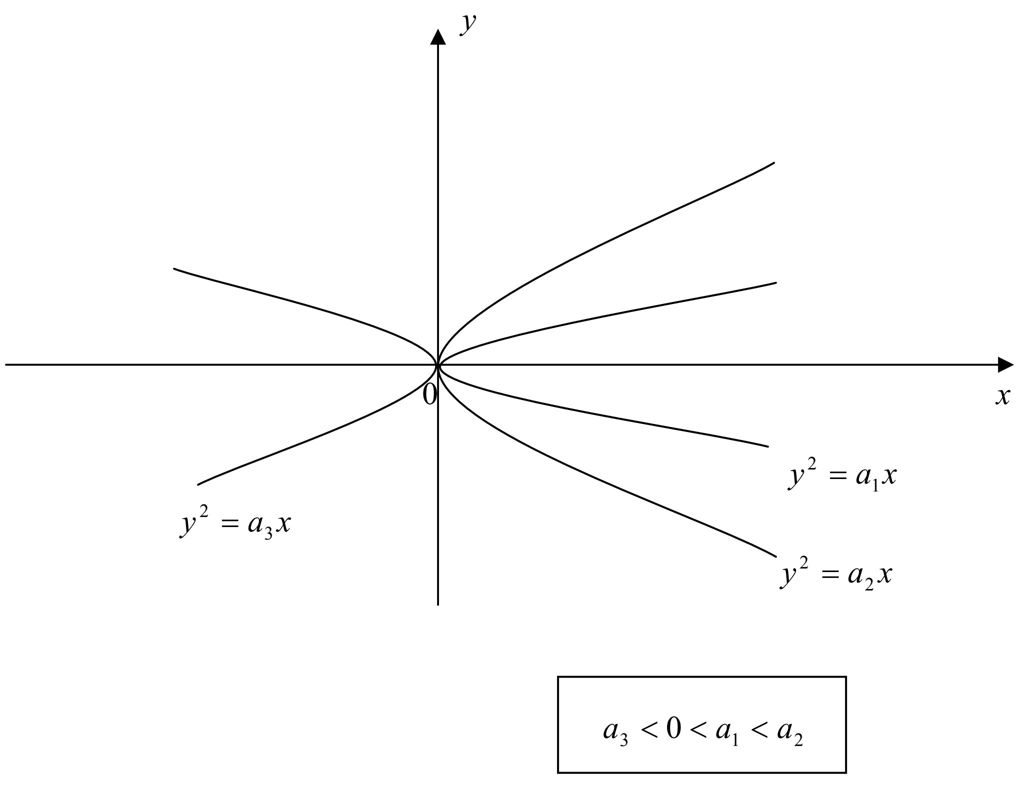


Note: The turning points $(1 + \sqrt{3},\ 2 + 2\sqrt{3})$ and $(1 - \sqrt{3},\ 2 - 2\sqrt{3})$ can be obtained by setting $\dfrac{dy}{dx} = 0$. This graph has no intersection with the axes.

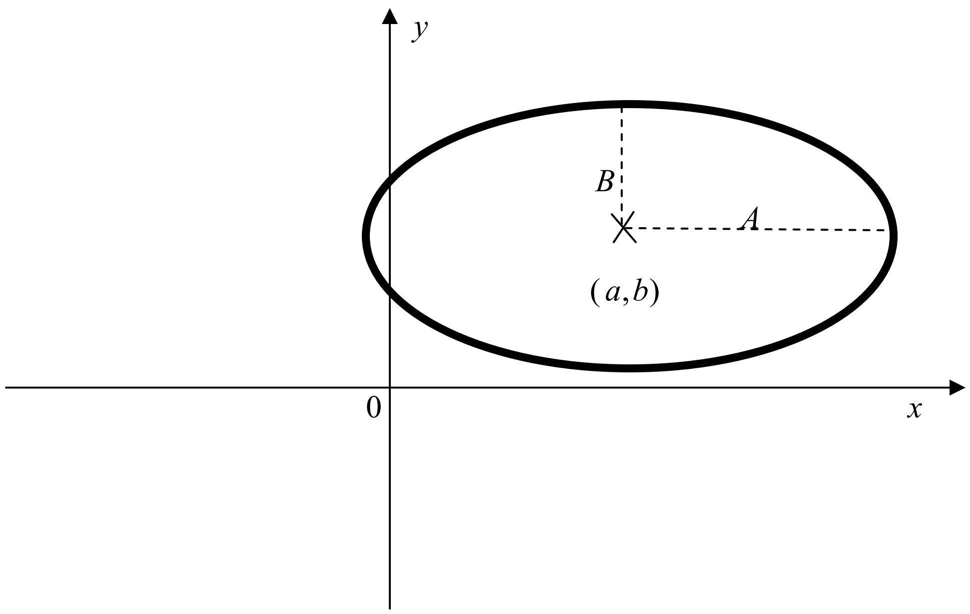
3. Graphs of the form $y = ax^2$ or $y^2 = ax$ (Parabolas)



Note that while the graph is symmetrical about the y-axis, this symmetry can be altered via a transformation, eg $y = a(x-5)^2$ is now symmetrical about the line x=5.



4. Graphs of the form $\frac{(x-a)^2}{A^2}+\frac{(y-b)^2}{B^2}=1$ (Ellipses)



Note that if $A = B$, then the ellipse becomes a **circle** centered at (a,b) with radius A units. When an ellipse is presented in a quadratic form, **completing the square** is required to fashion the equation into the structure above for extraction of its relevant characteristics.

Example: $9x^2-54x+4y^2+32y+109=0$

$9(x^2-6x)+4(y^2+8y)+109=0$

$9(x-3)^2-81+4(y+4)^2-64+109=0$

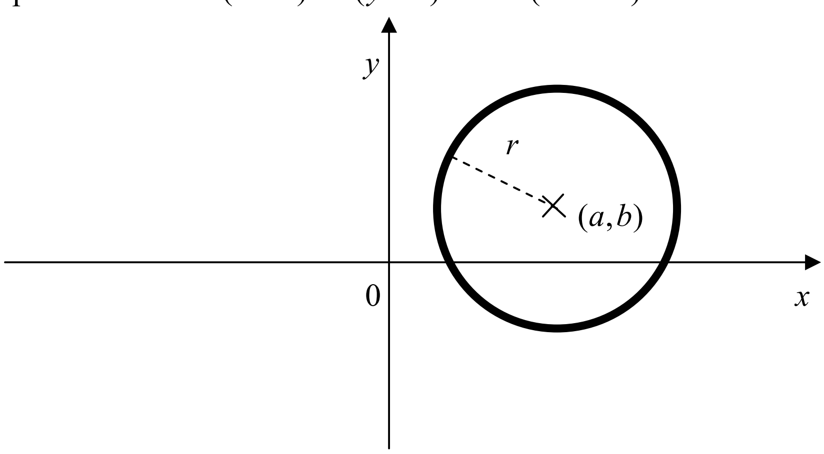
$9(x-3)^2+4(y+4)^2-36=0$

$9(x-3)^2+4(y+4)^2=36$

$\frac{(x-3)^2}{4}+\frac{(y+4)^2}{9}=1$ (Divide both sides by 36)

$\frac{(x-3)^2}{2^2}+\frac{(y+4)^2}{3^2}=1$

5. Graphs of the form $(x-a)^2+(y-b)^2=r^2$ (Circles)



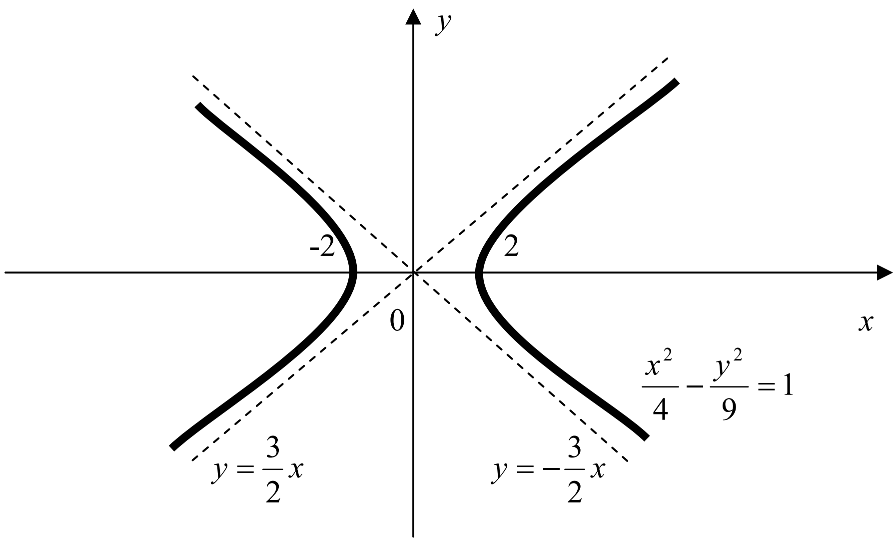
6. Graphs of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Hyperbolas)

Example: $\frac{x^2}{4}-\frac{y^2}{9}=1$

When $x=0$, $y^2=-9 \Rightarrow$ there are **no** y-intercepts.
When $y=0$, $x^2=4 \to x=\pm 2$
Hence, the graph has 2 x-intercepts at (2,0) and (-2,0)

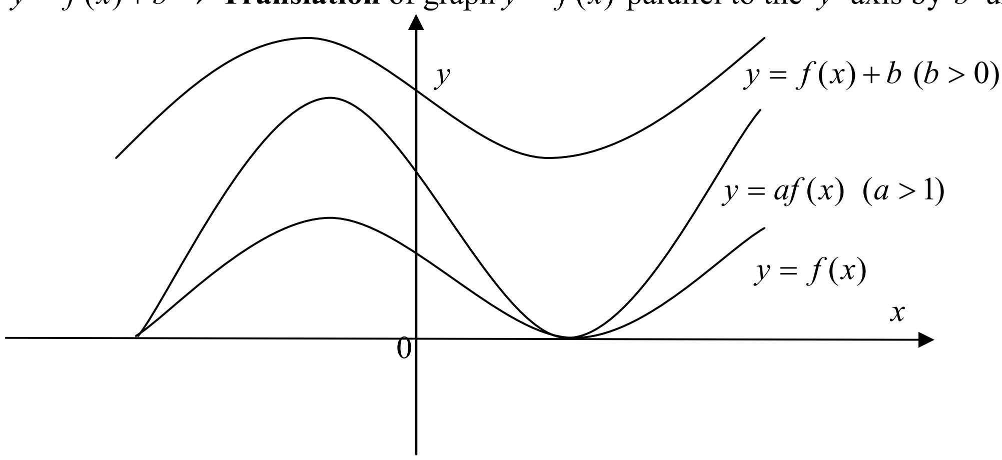
$$\frac{x^2}{4}-\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=\frac{x^2}{4}-1$$

When $x \to \pm\infty$, $\frac{y^2}{9} \to \frac{x^2}{4} \Rightarrow y=\pm\frac{3}{2}x$ are **oblique asymptotes**.



B. Transformation of graphs

1. Operations on y-coordinates:
Considering the original graph $y=f(x)$,
$y=af(x) \to$ **Scaling** of graph $y=f(x)$ parallel to the y-axis by a factor of a.
$y=f(x)+b \to$ **Translation** of graph $y=f(x)$ parallel to the y-axis by b units.

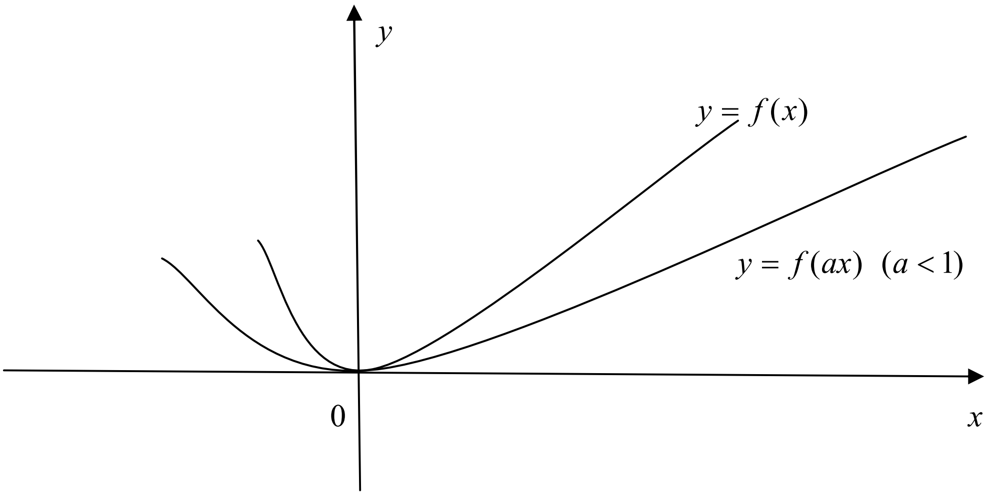


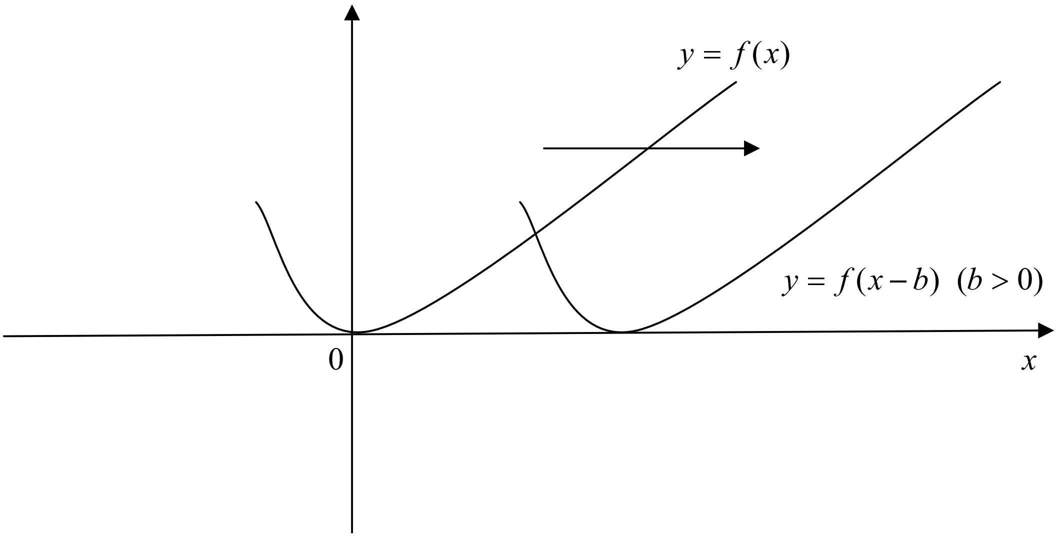
2. Operations on the x-coordinates:

Considering the original graph $y = f(x)$,

$y = f(ax) \rightarrow$ **Scaling** of graph $y = f(x)$ parallel to the x-axis by a factor of $\frac{1}{a}$.

$y = f(x-b) \rightarrow$ **Translation** of graph $y = f(x)$ parallel to the x-axis by b units





Note: for (1) and (2), both a and b can **assume the set of real values** $\Re$, although specific instances of the various graph transformations (eg for $y = af(x)$, only $a > 1$ was considered) were illustrated due to space constraints. A combination of transformations can exist as well, for example, $y = f[a(x-b)]$ implies the graph is scaled parallel to the x-axis by a factor of $\frac{1}{a}$, and subsequently translated horizontally along the x-axis by b units.

3. Miscellaneous transformations:

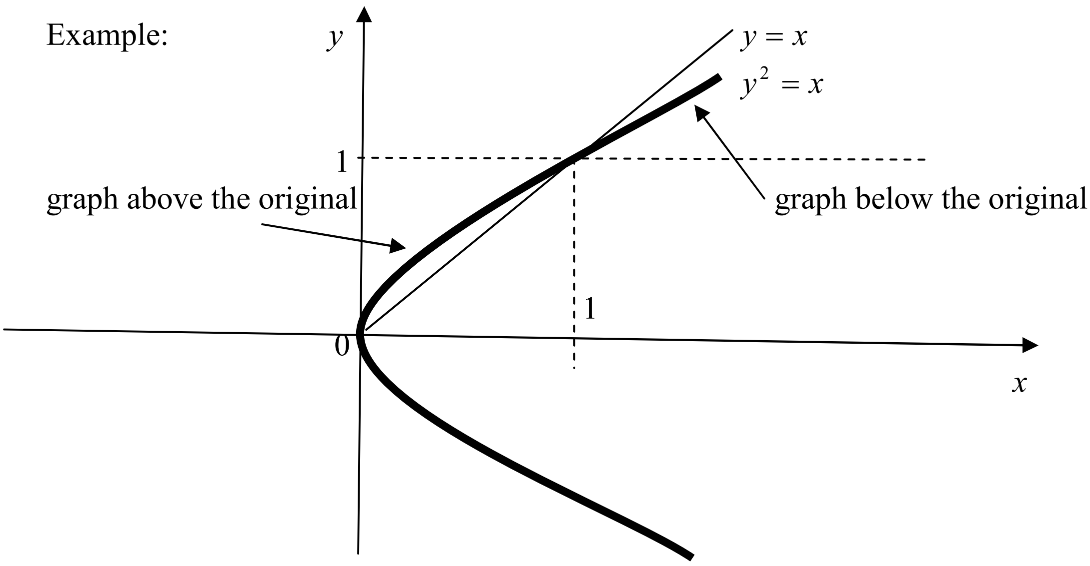
Considering the original graph $y = f(x)$,

$y = |f(x)| \rightarrow$ Reflection of all graph segments below the x-axis about the x-axis, keeping all other segments unchanged.

$y = f(|x|) \rightarrow$ **Erasure** of graph segment for $x < 0$, followed by a **reflection** of the graph segment for $x > 0$ about the y-axis.

To obtain $y^2 = f(x)$,
(i) Erase all graph segments below the y-axis.
(ii) For the graph segment above the y-axis, draw a **guiding line** $y = 1$.
(iii) All points with y-coordinates =0 or 1 will remain invariant (unchanged).
(iv) The new graph will exist above the original for $y < 1$, and subsequently below the original for $y > 1$
(v) Reflect the resulting graph about the x-axis.

Example:



To obtain $y = \dfrac{1}{f(x)}$,

(i) All $x-$intercepts will become **vertical asymptotes**, and vice-versa.
(ii) All **maximum** points will become **minimum** points, and vice-versa.
(iii) Graph segments which were decreasing with x will now **increase** with x, and vice versa.
(iv) All y values shall be **inverted**, with the exception of $x-$intercepts.
(v) Graph segments that were originally above the y-axis shall remain in the same region; this applies to graph segments below the y-axis as well.

Example: