

The idea to evaluate integrals involving modulus is as such:

For example, considering $\int_a^c |f(x)| dx$, we must first know what range of values of x for which $f(x)$ is negative, in this case lets assume it to be $f(x) \leq 0$ for $a \leq x \leq b$, where $b < c$.

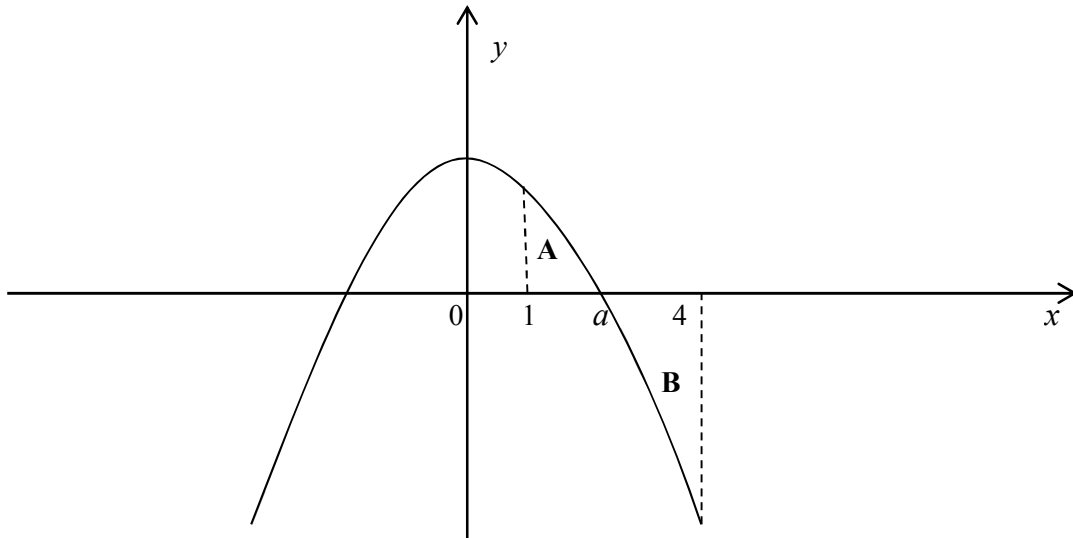
$$\text{Then } \int_a^c |f(x)| dx = -\int_a^b f(x) dx + \int_b^c f(x) dx$$

We section the integral into two parts, appending an additional negative sign to the integral part for which $f(x) \leq 0$, while simply integrating the remaining second part normally without the modulus sign.

With that, examine the following two problems:

(a) Solving $\int_1^4 |a^2 - x^2| dx$

For $y = a^2 - x^2$, where $1 < a < 4$, the graph is given below:



Noting that $a^2 - x^2 \geq 0$ for $1 \leq x \leq a$ and $a^2 - x^2 \leq 0$ for $a \leq x \leq 4$,

$$\begin{aligned} \int_1^4 |a^2 - x^2| dx &= \int_1^a a^2 - x^2 dx - \int_a^4 a^2 - x^2 dx \\ &= \left[a^2 x - \frac{x^3}{3} \right]_1^a - \left[a^2 x - \frac{x^3}{3} \right]_a^4 = \left(\frac{2a^3}{3} - a^2 + \frac{1}{3} \right) - \left(4a^2 - \frac{64}{3} - \frac{2a^3}{3} \right) \end{aligned}$$

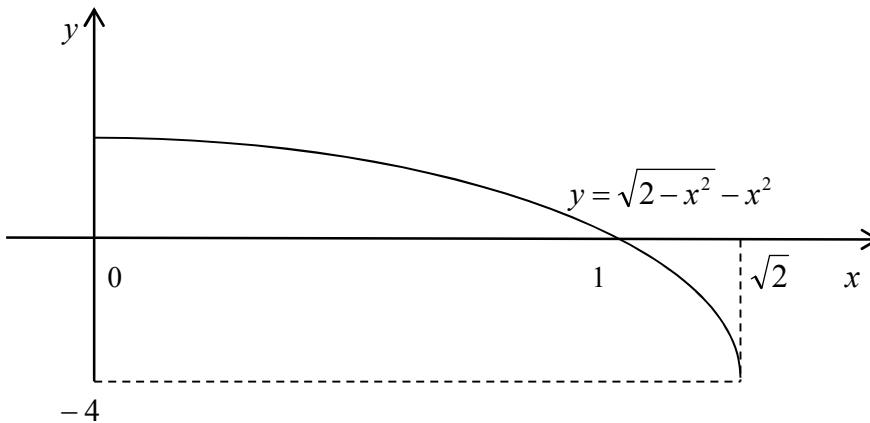
$$= \frac{4a^3}{3} - 5a^2 + \frac{65}{3} \quad (\text{shown})$$

$\int_1^4 |a^2 - x^2| dx$ can also be interpreted as **finding the sum of the areas** of regions A and B marked

on the above graph.

(b) Solving $\int_0^{\sqrt{2}} |\sqrt{2-x^2} - x^2| dx$

For $y = \sqrt{2-x^2} - x^2$, the graph looks like this for $0 \leq x \leq \sqrt{2}$:



$$\begin{aligned} \text{Hence, } \int_0^{\sqrt{2}} |\sqrt{2-x^2} - x^2| dx &= \int_0^1 \sqrt{2-x^2} - x^2 dx - \int_1^{\sqrt{2}} \sqrt{2-x^2} - x^2 dx \\ &= 0.9521 - (-0.3241) = 1.2762 \quad (\text{shown}) \end{aligned}$$

(Note: The above definite integrals were obtained using the GC, but if you wish to manually

integrate them to obtain exact values, to handle the $\sqrt{2-x^2}$ component, you will have to use the substitution $x = \sqrt{2} \sin \theta$.)