Differentiation Techniques Summary

Function	Derivative
ax^n	nax^{n-1}
$ax^n + bx^m$	$nax^{n-1} + bmx^{m-1}$
$(ax+b)^n$	$an(ax+b)^{n-1}$
$\left[f(x)\right]^n$	$n\left[f(x)\right]^{n-1}\left[f'(x)\right]$
$\sin\left[f(x)\right]$	$\left[f'(x)\right]\cos\left[f(x)\right]$
$\cos\left[f(x)\right]$	$-\left[f'(x)\right]\sin\left[f(x)\right]$
$\tan\left[f(x)\right]$	$\left[f'(x)\right]\sec^2\left[f(x)\right]$
$\operatorname{cosec}\left[f(x)\right]$	$-\left[f'(x)\right]\operatorname{cosec}\left[f(x)\right]\cot\left[f(x)\right]$
$\sec\left[f(x)\right]$	$\left[f'(x)\right]\sec\left[f(x)\right]\tan\left[f(x)\right]$
$\cot\left[f(x)\right]$	$-\left[f'(x)\right]\operatorname{cosec}^2\left[f(x)\right]$
$\sin^{-1}\left[f(x)\right]$	$\dfrac{\left[f'(x)\right]}{\sqrt{1-\left[f(x)\right]^2}}$
$\cos^{-1}\left[f(x)\right]$	$-\dfrac{\left[f'(x)\right]}{\sqrt{1-\left[f(x)\right]^2}}$
$\tan^{-1}\left[f(x)\right]$	$\dfrac{\left[f'(x)\right]}{1+\left[f(x)\right]^2}$

Function	Derivative
$a^{f(x)}$	$a^{f(x)}(\ln a)f'(x)$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

Product Rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Implicit Differentiation:

$$\frac{d}{dx}f(y) = \frac{d}{dy}f'(y) \bullet \frac{dy}{dx}$$

<u>Knowledge of proving specific differential identities:</u>

1. Show that $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

   Let $y = \tan^{-1} x$, then $\tan y = x$

   Differentiating both sides *wrt* x gives $\sec^2 y\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

(Similar approaches shall be taken for proving the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$)

2. Show that $\dfrac{d}{dx}(a^x) = (\ln a)(a^x)$

   Let $y = a^x$, then $\ln y = x\ln a$

Differentiating both sides *wrt* x gives $\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = \ln a \Rightarrow \frac{dy}{dx} = y(\ln a)$

$$\therefore \quad \frac{dy}{dx} = (a^x)(\ln a) \qquad \text{(shown)}$$

Note that many other variations can surface within the examinations, where techniques like implicit differentiation, product rule or quotient rule may have to be employed.

<u>Manipulation of derivatives to achieve targeted differential equations:</u>

Example: If $y = e^x \ln x$, (a) Find $\frac{dy}{dx}$.

(b) Show that $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - (1+x)y = 2e^x$

<u>SOLUTIONS :</u>

$$\frac{dy}{dx} = e^x\left(\frac{1}{x}\right) + e^x \ln x = e^x\left(\frac{1}{x}\right) + y$$

$$\frac{d^2y}{dx^2} = e^x\left(\frac{1}{x}\right) - e^x\left(\frac{1}{x^2}\right) + \frac{dy}{dx}$$

$$x\frac{d^2y}{dx^2} = e^x - e^x\left(\frac{1}{x}\right) + x\frac{dy}{dx} = e^x - e^x\left(\frac{1}{x}\right) + e^x + xy$$

$$x\frac{d^2y}{dx^2} + e^x\left(\frac{1}{x}\right) - xy = 2e^x$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y - xy = 2e^x$$

$$\therefore \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} - (1+x)y = 2e^x \quad \text{(shown)}$$