Complex Numbers Summary

1. Cartesian representation and laws:

$z = x + iy \rightarrow \ \mathrm{Re}(z) = x, \ \mathrm{Im}(z) = y, \ |z| = \sqrt{x^2 + y^2}, \ z^* = x - iy$

(i) $z + z^* = 2\,\mathrm{Re}(z)$
(ii) $z - z^* = 2i\,\mathrm{Im}(z)$
(iii) $zz^* = |z|^2$
(iv) $(z \pm w)^* = z^* \pm w^*$, where w is also a complex number
(v) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ where $-\pi < \arg(z_1), \ \arg(z_2) \leq \pi$
(vi) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note: If a polynomial $F(z) = 0$ has real coefficients throughout its entire structure, then complex roots (if any) **MUST** occur in conjugate pairs.

2. Polar representation and laws:

$z = r(\cos\theta + i\sin\theta)$, where $r = |z| > 0$, $\arg(z) = \theta$ and $-\pi < \theta \leq \pi$

De Moivre's Theorem:
If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$

Useful information:
(i) $\cos(-\theta) = \cos\theta$; $\sin(-\theta) = -\sin\theta$
(ii) If $z = r(\cos\theta + i\sin\theta)$, then $z^* = r(\cos\theta - i\sin\theta)$
(iii) $z + z^* = 2r\cos\theta$, $z - z^* = 2r(i\sin\theta)$
(Note in particular if $r = 1$, then $z^* = \cos\theta - i\sin\theta = \dfrac{1}{z}$)

3. Euler's(exponential) representation and laws:

$z = re^{i\theta}$, where $r = |z| > 0$, $\arg(z) = \theta$ and $-\pi < \theta \leq \pi$

(i) $\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

(ii) $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

(iii) $\left(re^{i\theta}\right)^n = re^{in\theta}$

Useful information:
(i) If $z = re^{i\theta}$, then $z^* = re^{-i\theta}$
(ii) $(z - re^{i\theta})(z - re^{-i\theta}) = z^2 - z(re^{i\theta} + re^{-i\theta}) + r^2 = z^2 - 2z\cos\theta + r^2$

4. Solving higher order polynomial equations:
(a) Roots of unity:

$$z^n = 1 \Rightarrow z^n = e^{i(2k\pi)}$$

$$\therefore \; z = e^{i\left(\frac{2k\pi}{n}\right)}, \qquad n = 0, 1, 2, 3, \ldots\ldots.. n-1$$

Note: Ensure that the argument of each single individual complex root is presented in the standard range required, ie $-\pi < \arg(z) \leq \pi$.

(b) Solving general polynomial equations:

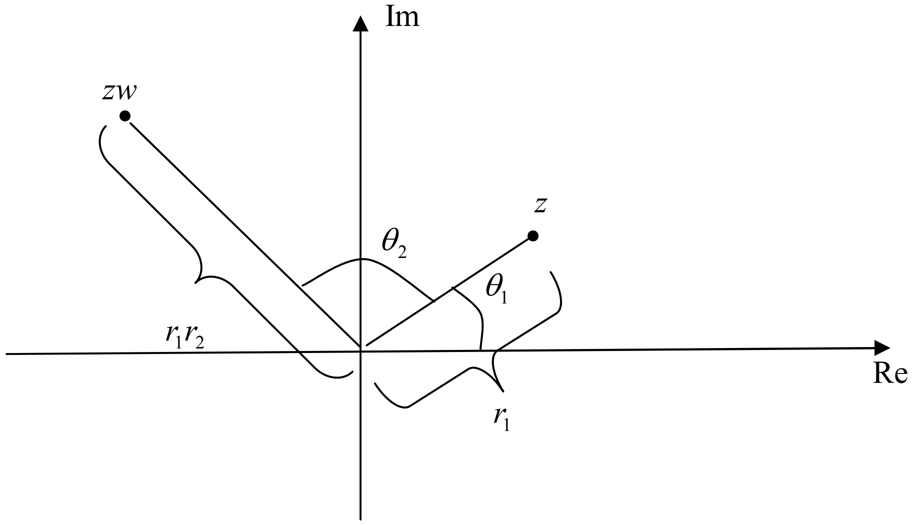
$z^n = F(z) \Rightarrow z^n = re^{i(2k\pi+\theta)}$, where $F(z)$ has been transformed into the Euler's representation and its argument θ is added to a **mandatory cycling factor** $2k\pi$

$$\therefore z = e^{i\left(\frac{2k\pi+\theta}{n}\right)} \quad n = 0, 1, 2, 3, \ldots\ldots.. n-1$$

Note: Ensure that the argument of each single individual complex root is presented in the standard range required, ie $-\pi < \arg(z) \leq \pi$.

5. Physical implications of multiplying one complex number by another:

Consider a complex number $z = r_1 e^{i\theta_1}$. If it is multiplied by another complex number $w = r_2 e^{i\theta_2}$, then the physical effect is that of changing the length of the cord **joining** z **and the origin** (in the Argand diagram)by a factor of r_2, and subsequently rotating this cord by an angle of θ_2. The resultant complex number is therefore $r_1 r_2 e^{i(\theta_1+\theta_2)}$.

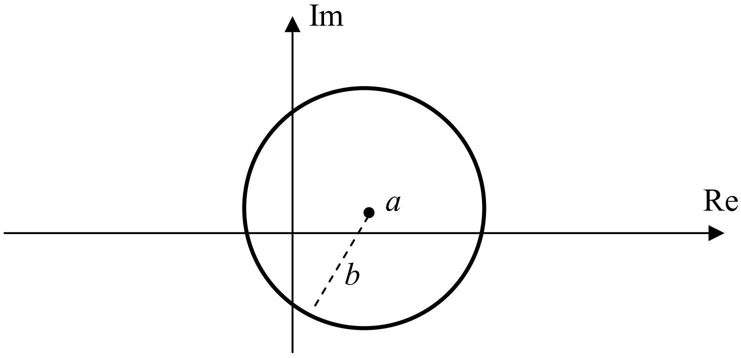


6. Loci of various complex number constructs

(i) $|z - a| = b$

Interpretation: Set of variable points denoted by z which are always b units away from a fixed complex number a.

Locus: A circle centered at complex number a with radius b

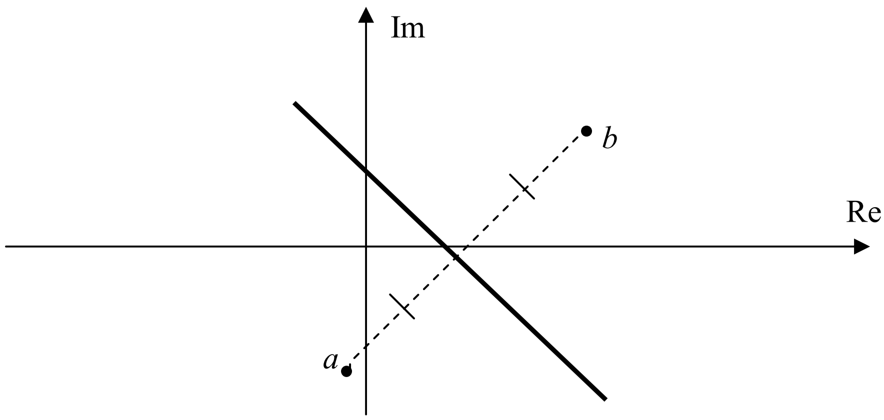


Note: If $|z-a| \leq b$, then the locus shall be the entire interior of the circle coupled with the boundary of the circle itself. Similar logic must be applied to varying inequalities for this circular loci construct as well.

(ii) $|z-a|=|z-b|$

Interpretation: Set of variable points denoted by z which are equidistant from two unique, fixed complex numbers a and b.

Locus: A line that bisects the cord joining complex numbers a and b in a perpendicular fashion



(iii) $\arg(z-a)=\theta$

Interpretation: Set of variable points denoted by z which will form an argument of θ around a fixed complex number a.

Locus: A line that is pivoted at a and possesses a standard argument θ.