

Arithmetic and Geometric Progressions (AP/GP) Summary

1. Arithmetic Progression (AP) :

- characterized by a common difference d

n th term $T_n = a + (n-1)d$, where a is the first term of the series

Sum to n terms $S_n = \frac{n}{2}(a+l)$, where l denotes the last term of the series of n terms

or $S_n = \frac{n}{2}[2a + (n-1)d]$ (to be used if value of l is not readily available)

2. Geometric Progression (GP) :

- characterized by a common ratio r

n th term $T_n = ar^{n-1}$, where a is the first term of the series

Sum to n terms $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$

Note: If r is negative, then the sequence comprises terms alternating between positive and

negative ones. Also, if $|r| < 1$, then the series is a **convergent** one, ie $T_n \rightarrow k$ (k is a real

value) as $n \rightarrow \infty$, and it possesses a **sum to infinity** given by $S_\infty = \frac{a}{1-r}$.

3. Techniques relevant to problem solving:

(i) Forming an equation relating three terms of a sequence.

If x , y and z are consecutive terms of an AP, then we have $y - x = z - y$

If x , y and z are consecutive terms of a GP, then we have $\frac{z}{y} = \frac{y}{x}$

(ii) Recognising that any uninterrupted part of a sequence will retain the main characteristics of its parent sequence:

Example: A descending geometric series has first term a and the common ratio r is positive.

The sum of the first 5 terms is twice the sum of terms from the 6th to 15th inclusive.

While it is not erroneous to state that $S_5 = 2(S_{15} - S_5) \Rightarrow \frac{a(r^5 - 1)}{r - 1} = 2 \left[\frac{a(r^{15} - 1)}{r - 1} - \frac{a(r^5 - 1)}{r - 1} \right]$,

it would be far more elegant to use the understanding that the 6th to 15th (10 terms altogether) terms of the GP can be construed as a separate GP series with first term $= ar^{6-1} = ar^5$ but with the same common ratio r as the parent series. Hence, we have

$$\frac{a(r^5 - 1)}{r - 1} = 2 \left[\frac{ar^5(r^{10} - 1)}{r - 1} \right],$$

which would render future developments of any solution a tad more efficient.

(iii) The bank interest problem (and its parallel storylines):

Example: At the end of a month, a customer owes a bank \$1500. In the middle of the month, the customer pays \$ x to the bank where $x < 1000$, and at the end of the month the bank adds interest at a rate of 4% of the remaining amount still owed. This process continues every month until the money owed is repaid in full.

- (i) Find the value of x for which the customer still owes \$1500 at the start of every month.
- (ii) Find the value of x for which the whole amount owed is paid off exactly after the second payment.
- (iii) Show that the value of x for which the whole amount owed is paid off exactly after the $(n + 1)$ th payment is given by

$$x = \frac{1500r^n(r - 1)}{r^{n+1} - 1}, \text{ where } r = 1.04$$

SOLUTIONS :

(i) $1500 = (1500 - x)(1.04) \Rightarrow x = \57.69 (shown)

(ii) After 1st payment of \$ x , the amount owed = $(1500 - x)(1.04)$

Therefore, $(1500 - x)(1.04) = x \Rightarrow x = \764.71 (shown)

(iii) After the second payment of \$ x , amount owed at beginning of 3rd month is

$$[(1500 - x)1.04 - x](1.04) = 1500(1.04)^2 - 1.04^2x - 1.04x$$

After the second payment of \$x, amount owed at beginning of 4th month is

$$[1500(1.04)^2 - 1.04^2 x - 1.04x - x](1.04)$$

$$= 1500(1.04)^3 - 1.04^3 x - 1.04^2 x - 1.04x$$

After n th payment of \$x, the amount still owed at the beginning of the $(n+1)$ th month

$$= 1500(1.04)^n - 1.04^n x - 1.04^{n-1} x - \dots - 1.04^2 x - 1.04x$$

$$= 1500r^n - x(r + r^2 + \dots + r^n) \quad \text{where } r = 1.04$$

At the $(n+1)$ th payment,

$$x = 1500r^n - x(r + r^2 + \dots + r^n)$$

$$x(1 + r + r^2 + \dots + r^n) = 1500r^n$$

$$\Rightarrow x \left(\frac{r^{n+1} - 1}{r - 1} \right) = 1500r^n \Rightarrow x = \frac{1500r^n (r - 1)}{r^{n+1} - 1} \quad (\text{shown})$$

Strategies to note in the above solution:

- (a) Consolidation of values is highly discouraged when seeking out patterns in expressions, eg for (iii) of the immediate above problem, leave $1500(1.04)^2$ in its raw form, and do not simplify it as 1622.4.
- (b) Exercise patience and write a few iterations applicable to various time frames, ie the outstanding amounts say after 1 month, 2 months, 3 months, etc or 1 year, 2 years, 3 years etc depending on the question context. From there, attempt to identify a trend/pattern within these formulations.
- (c) Be acutely aware of the exact occurrences of events in reference to the timeline of the question, eg whether interest is appended at the beginning or end of a designated period, whether payment is made in the middle or the end of the period etc.

(iv) Transforming a GP into an AP (or vice versa):

This usually includes index operations or inclusion of logarithmic changes to every existing term of the original series.

Example: A geometric series $\{x_n\}$ has first term a and common ratio r . The sequence of numbers $\{y_n\}$ satisfies the relation $y_n = \log_3 x_n$ for $n \in \mathbb{Z}, n \geq 1$.

- (i) If the product of x_5 and x_{16} is 81, find the value of $\sum_{k=1}^{20} \log_3 x_k$.
(ii) Show that $\{y_n\}$ is an arithmetic sequence.

SOLUTIONS :

(i) $x_5 = ar^4, x_{16} = ar^{15}$

$$x_5 x_{16} = 81 \Rightarrow a^2 r^{19} = 81$$

$$\begin{aligned} \sum_{k=1}^{20} \log_3 x_k &= \log_3 x_1 + \log_3 x_2 + \log_3 x_3 + \dots + \log_3 x_{20} \\ &= \log_3 (x_1 x_2 x_3 \dots x_{20}) \\ &= \log_3 [(x_1 x_{20})(x_2 x_{19}) \dots (x_{10} x_{11})] \\ &= \log_3 [(81)(81) \dots (81)] \\ &= \log_3 81^{10} = 10 \log_3 81 = 10(4) = 40 \text{ (shown)} \end{aligned}$$

(Note: $x_1 x_{20} = x_2 x_{19} = \dots = x_{10} x_{11} = a^2 r^{19} = 81$)

(ii) $y_n - y_{n-1} = \log_3 x_n - \log_3 x_{n-1} = \log_3 \left(\frac{x_n}{x_{n-1}}\right) = \log_3 r$, which is a constant.

Hence, $\{y_n\}$ is an arithmetic sequence. (shown)