

## Applications of Differentiation Summary

### 1. Parametric Equations/Tangents/Normals:

For a set of parametric descriptions whereby  $x = f(t)$ ,  $y = g(t)$ ,

$$\boxed{\frac{dy}{dx} = \frac{g'(t)}{f'(t)}}$$

Equation of **tangent**:  $y - g(t) = \frac{g'(t)}{f'(t)}[x - f(t)]$

Equation of **normal**:  $y - g(t) = -\frac{f'(t)}{g'(t)}[x - f(t)]$

#### Question types:

Example: A curve has parametric equations

$$x = 1 + 2 \sin \theta, \quad y = 4 + \cos \theta.$$

P is a point on the curve where  $\theta = \frac{\pi}{6}$ . Find the area of the triangle bounded by the tangent and normal at P, as well as the y-axis.

#### SOLUTIONS :

At point P,  $x = 2$ ,  $y = 4 + \frac{\sqrt{3}}{2}$

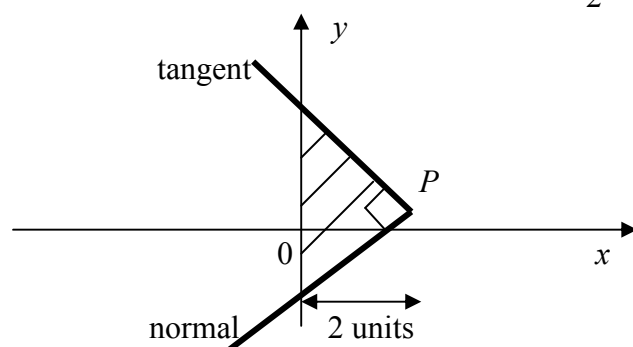
$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta \Rightarrow \text{At P, } \frac{dy}{dx} = -\frac{1}{2\sqrt{3}}$$

Equation of tangent:  $y - (4 + \frac{\sqrt{3}}{2}) = -\frac{1}{2\sqrt{3}}(x - 2)$

When it cuts the *y*-axis,  $x = 0 \Rightarrow y = 4 + \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}$

Equation of normal:  $y - (4 + \frac{\sqrt{3}}{2}) = 2\sqrt{3}(x - 2)$

When it cuts the *y*-axis,  $x = 0 \Rightarrow y = 4 - \frac{7\sqrt{3}}{2}$



$$\text{Area of triangle} = \frac{1}{2}(2) \left[ \left( 4 + \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \right) - \left( 4 - \frac{7\sqrt{3}}{2} \right) \right] = 4\sqrt{3} + \frac{1}{\sqrt{3}} = \frac{13}{\sqrt{3}} \text{ units}$$

(shown)

Example: A curve is defined by the parametric equations  $x = t^2, y = t^3$ . Show that the equation of the tangent to the curve at the point P ( $p^2, p^3$ ) is  $2y - 3px + p^3 = 0$ . Show that there is just one point on the curve at which the tangent passes through the point (-3,-5), and determine the coordinates of this point.

**SOLUTIONS :**

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

At P,  $\frac{dy}{dx} = \frac{3}{2}p$

Equation of tangent is  $y - p^3 = \frac{3}{2}p(x - p^2)$

Rearranging gives  $2y - 3px + p^3 = 0$  (shown)

Since it passes through (-3,-5),

$$2(-5) - 3p(-3) + p^3 = 0 \Rightarrow -10 + 9p + p^3 = 0$$

$$(p - 1)(p^2 + p + 10) = 0$$

$\therefore p = 1$  and point is  $(1^2, 1^3) = (1, 1)$  (shown)

(Quadratic polynomial has no real roots since determinant  $= b^2 - 4ac = -39 < 0$ )

**2. Rates of Change:**

The chain rule is typically used in such questions; variations of this rule include the following:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}, \quad \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}, \quad \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \text{ etc}$$

Where  $V, A, r, h$  typically denote volume, area, radius and height quantities respectively. Ensure **knowledge of the separate derivatives are thorough and clear**, eg  $\frac{dV}{dt}$  describes the rate of change of volume.

**Popular formulas:**

Volume of Cone  $-\frac{1}{3}\pi(r^2)h$  ; Volume of Sphere  $-\frac{4}{3}\pi(r^3)$ ;

Surface area of Sphere  $-4\pi(r^2)$

**Question types:**

Example: An inverted cone of base radius 4cm and height 8cm is initially filled with

water. Water drips out from the vertex at a rate of  $2\pi \text{ cm}^3 \text{ s}^{-1}$ . Find the rate of decrease in the depth of the water in the cone 16 seconds after the dripping starts.

**SOLUTIONS :**

$$\frac{r}{h} = \frac{4}{8} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h$$

At any time t, volume remaining  $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2(h) = \frac{1}{12}\pi h^3$

Differentiating V with respect to h,  $\frac{dV}{dh} = \frac{\pi}{4}h^2$

When t=16,  $\frac{1}{12}\pi h^3 = \frac{1}{3}\pi(4)^2(8) - 2(16) \Rightarrow h^3 = 128$

By the chain rule,  $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \Rightarrow -2\pi = \frac{\pi}{4}(\sqrt[3]{128})^2 \times \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = -0.315 \text{ cms}^{-1} \text{ (shown)}$$

**3. Maxima/Minima problems:**

Maxima/minima problems typically employ the concept of **stationary values**, and ascertaining the nature of those values (ie maximum or minimum) via the **sign test** or **investigating the second derivative** if it is reasonably easy to attain.

**Question types:**

Example:

A length of channel of given depth d is to be made from a rectangular sheet of metal of width 2a. The metal is to be bent in such a way that the cross section ABCD is as shown in the figure, with  $AB+BC+CD=2a$  and with AB and CD inclined to the line BC at an angle  $\theta$ .



Show that  $BC=2(a - d \operatorname{cosec}\theta)$  and that the area of the cross section ABCD is  $2ad + d^2(\cot\theta - 2 \operatorname{cosec}\theta)$ .

Show that the maximum value of  $2ad + d^2(\cot\theta - 2 \operatorname{cosec}\theta)$ , as  $\theta$  varies,

is  $d(2a - d\sqrt{3})$ .

By considering the length of BC, show that the cross sectional area can only be made equal to this maximum value if  $2d \leq a\sqrt{3}$ .

**SOLUTIONS :**

$$\sin \theta = \frac{d}{AB} \Rightarrow AB = CD = d \operatorname{cosec} \theta$$

$$BC = 2a - AB - CD = 2a - 2d \operatorname{cosec} \theta = 2(a - d \operatorname{cosec} \theta) \quad (\text{shown})$$

Area of cross section ABCD

$$= 2d(a - d \operatorname{cosec} \theta) + 2 \left[ \frac{1}{2} (d)(d \cot \theta) \right] = 2d(a - d \operatorname{cosec} \theta) + d(d \cot \theta)$$

$$= 2ad + d^2(\cot \theta - 2 \operatorname{cosec} \theta) \quad (\text{shown})$$

$$\text{Let } y = 2ad + d^2(\cot \theta - 2 \operatorname{cosec} \theta)$$

$$\frac{dy}{d\theta} = d^2[-\operatorname{cosec}^2 \theta + 2 \cot \theta \operatorname{cosec} \theta]$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \operatorname{cosec}^2 \theta = 2 \cot \theta \operatorname{cosec} \theta \rightarrow \operatorname{cosec} \theta = 2 \cot \theta$$

$$\therefore \cos \theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}$$

$$\text{Maximum value} = 2ad + d^2 \left[ \frac{1}{\sqrt{3}} - 2 \left( \frac{2}{\sqrt{3}} \right) \right] = d(2a - d\sqrt{3}) \quad (\text{shown})$$

$$BC = 2(a - d \operatorname{cosec} \theta), \text{ putting in } \theta = \frac{\pi}{3},$$

$$BC = 2 \left( a - \frac{2}{\sqrt{3}} d \right)$$

$$\text{Since } BC \geq 0, \quad a \geq \frac{2}{\sqrt{3}} d \Rightarrow 2d \leq \sqrt{3}a \quad (\text{shown})$$

Example:

The equation of a curve is  $y = ax^2 - 2bx + c$ , where a, b and c are constants,  $a > 0$ . Find, in terms of a, b and c, the coordinates of the turning point on the curve, and state whether it is a maximum or minimum point.

**SOLUTIONS :**

$$y = ax^2 - 2bx + c \rightarrow \frac{dy}{dx} = 2ax - 2b$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{b}{a}, \quad y = c - \frac{b^2}{a}$$

$$\frac{d^2y}{dx^2} = 2a > 0 \quad [\because a > 0] \quad \text{Hence, } \left( \frac{b}{a}, c - \frac{b^2}{a} \right) \text{ is a minimum point.}$$