Additional Vectors Problems Solutions

1(i) (a) Shortest distance from origin to plane $\prod_1 = \dfrac{4}{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\right|} = \dfrac{4}{3}$ units

Hence, foot of perpendicular is given by $\dfrac{4}{3}\hat{n}_1 = \dfrac{4}{3}\left(\dfrac{1}{3}\right)\begin{pmatrix}2\\1\\2\end{pmatrix} = \dfrac{4}{9}\begin{pmatrix}2\\1\\2\end{pmatrix}$ (shown)

(b) Equation of line l: $r = \begin{pmatrix}0\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}\lambda\\2+2\lambda\\1-2\lambda\end{pmatrix}$

$$\begin{pmatrix}\lambda\\2+2\lambda\\1-2\lambda\end{pmatrix} \bullet \begin{pmatrix}2\\1\\2\end{pmatrix} = 2\lambda + 2 + 2\lambda + 2 - 4\lambda = 4$$

Therefore, l **lies in the plane**. (shown)

(ii) $n_2 = \begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}-6\\6\\3\end{pmatrix} // \begin{pmatrix}-2\\2\\1\end{pmatrix}$

$\therefore$ Equation of $\prod_2$ is $r \bullet \begin{pmatrix}-2\\2\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\1\end{pmatrix} \bullet \begin{pmatrix}0\\2\\1\end{pmatrix} = 5$ (shown)

(iii) $n_3 = \begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}-2\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\-6\\6\end{pmatrix} // \begin{pmatrix}1\\2\\-2\end{pmatrix}$

Let the foot of perpendicular from $\begin{pmatrix}0\\2\\1\end{pmatrix}$ to $\prod_3$ be A

Then $\overrightarrow{OA} - \begin{pmatrix}0\\2\\1\end{pmatrix} = 3\hat{n}_3 = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ or $\begin{pmatrix}0\\2\\1\end{pmatrix} - \overrightarrow{OA} = 3\hat{n}_3 = \begin{pmatrix}1\\2\\-2\end{pmatrix}$

ie $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

Possible equations of Π_3

is $r \bullet \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = 11$ or $r \bullet \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = -7$

The possible augmented matrices formed from the intersection of the 3 planes

are $\left(\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ -2 & 2 & 1 & 5 \\ 1 & 2 & -2 & 11 \end{array}\right)$ or $\left(\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ -2 & 2 & 1 & 5 \\ 1 & 2 & -2 & -7 \end{array}\right)$

Solving both of them give the possible points of intersection as $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ (shown)

2(a) Equation of line can be rewritten as $r = \begin{pmatrix} 1-3\lambda \\ 3-22\lambda \\ -7+42\lambda \end{pmatrix}$

Since, $\begin{pmatrix} 1-3\lambda \\ 3-22\lambda \\ -7+42\lambda \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 6-18\lambda+9-66\lambda-14+84\lambda = 1$ and

$\begin{pmatrix} 1-3\lambda \\ 3-22\lambda \\ -7+42\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = 2-6\lambda-18+132\lambda+21-126\lambda = 5,$

$r = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -22 \\ 42 \end{pmatrix}$ is the line of intersection of the planes P_1 and P_2. (shown)

(b) The normal to the required plane is given by $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 22 \\ -42 \end{pmatrix}$

(conversely, you can also arrive at the above by noting that the direction vector of the line of intersection of P_1 and P_2 is essentially denotes the vector normal to the required plane)

Equation of required plane is $r \bullet \begin{pmatrix} 3 \\ 22 \\ -42 \end{pmatrix} = \begin{pmatrix} 3 \\ 22 \\ -42 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = 363$ (shown)

(c) Since $n_1 \bullet n_2 = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = 12 - 18 - 6 = -12 < 0,$ the angle between the planes P_1 and P_2 is obtuse.

Hence, the direction vector of the acute angle bisector between the two planes P_1 and P_2 is given by

$$n_1 - n_2 = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

and the equation of this line is $r = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$ (shown)

(d) Let the plane bisecting the acute angle between planes P_1 and P_2 be P_3.

Then the normal to P_3 is given by $n_3 = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 22 \\ -42 \end{pmatrix} = \begin{pmatrix} -488 \\ 183 \\ 61 \end{pmatrix} // \begin{pmatrix} -8 \\ 3 \\ 1 \end{pmatrix}$

$\therefore$ Equation of P_3 is $r \bullet \begin{pmatrix} -8 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = -8 + 9 - 7 = -6$ (shown)

3(i) Area of triangle $ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|(b-a)\times(c-a)\right|$

$$= \frac{1}{2}\left|b\times c - b\times a - a\times c + a\times a\right| = \frac{1}{2}\left|b\times c + a\times b + c\times a\right| \text{ (shown)}$$

$[\because\ a\times a = \underset{\sim}{0}\,]$

(ii) Vector perpendicular to triangle $ABC = \overrightarrow{AB}\times\overrightarrow{AC} = b\times c + a\times b + c\times a$ (shown)

(iii) Denoting the normal to plane ABC as n,

Shortest distance from the vertex D to the plane ABC

$$= \left|(d-a)\bullet\hat{n}\right| = \frac{1}{\left|b\times c + a\times b + c\times a\right|}\left|(d-a)\bullet\left(b\times c + a\times b + c\times a\right)\right|$$

$$= \frac{1}{\left|b\times c + a\times b + c\times a\right|}\left|d\bullet\left(b\times c + a\times b + c\times a\right) - a\bullet\left(b\times c + a\times b + c\times a\right)\right|$$

$$= \frac{1}{\left|b\times c + a\times b + c\times a\right|}\left|d\bullet\left(b\times c + a\times b + c\times a\right) - a\bullet\left(b\times c\right)\right|$$

$\therefore$ Volume of tetrahedron $ABCD$ is

$$\frac{1}{3}\left[\frac{1}{2}\left|b\times c + a\times b + c\times a\right|\right]\left[\frac{1}{\left|b\times c + a\times b + c\times a\right|}\left|d\bullet\left(b\times c + a\times b + c\times a\right) - a\bullet\left(b\times c\right)\right|\right]$$

$$= \frac{1}{6}\left|d\bullet\left(b\times c + a\times b + c\times a\right) - a\bullet\left(b\times c\right)\right| \quad \text{(shown)}$$

4. A vector normal to plane $\prod_1$ is $n_1 = \begin{pmatrix}2\\0\\3\end{pmatrix}\times\begin{pmatrix}0\\-4\\5\end{pmatrix} = \begin{pmatrix}12\\-10\\-8\end{pmatrix} // \begin{pmatrix}6\\-5\\-4\end{pmatrix}$

A vector normal to plane $\prod_2$ is $n_2 = \begin{pmatrix}0\\3\\1\end{pmatrix}\times\begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}5\\-1\\3\end{pmatrix}$

Direction vector of line of intersection of $\prod_1$ and $\prod_2$ is $\begin{pmatrix}5\\-1\\3\end{pmatrix}\times\begin{pmatrix}6\\-5\\-4\end{pmatrix} = \begin{pmatrix}19\\38\\-19\end{pmatrix} // \begin{pmatrix}1\\2\\-1\end{pmatrix}$

Hence, equation of line of intersection l is $r = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (shown)

A vector normal to plane $\prod_3$ is $n_3 = \left[\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right] \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix}$$

Equation of $\prod_3$ is $r \bullet \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 18 - 8 + 30 = 40 \quad \Rightarrow 9x - 2y + 5z = 40$ (shown)

Equation of $\prod_1$ is $r \bullet \begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 12 - 20 - 24 = -32 \quad \Rightarrow 6x - 5y - 4z = -32$

Equation of $\prod_2$ is $r \bullet \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 10 - 4 + 18 = 24 \quad \Rightarrow 5x - y + 3z = 24$

Since the line l lies on all 3 planes, hence the system of equations

$$6x - 5y - 4z = -32$$

$$5x - y + 3z = 24$$

$$9x - 2y + 5z = 40$$

will have an infinite number of solutions given by $r = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (shown)

5. (a) Let the acute angle between l_1 and the $x-$axis be θ.

Then $\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left| \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| \cos\theta \rightarrow \cos\theta = \dfrac{2}{\sqrt{17}}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{2}{\sqrt{17}}\right) = 61^\circ$ (shown)

(b) Since l_2 passes through point A, $\begin{pmatrix} 4+a\lambda \\ 6+4\lambda \\ 2+9\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ b \end{pmatrix}$

By comparison, $6+4\lambda = -2 \Rightarrow \lambda = -2$

$4 + a\lambda = 2 \Rightarrow 4 - 2a = 2, \quad a = 1$

$2 + 9\lambda = b \Rightarrow b = 2 + 9(-2) = -16$ (shown)

(i) When l_1 and l_2 intersect,

$$\begin{pmatrix} 1+2\mu \\ 2\mu \\ -4-3\mu \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ 6+4\lambda \\ 2+9\lambda \end{pmatrix}$$

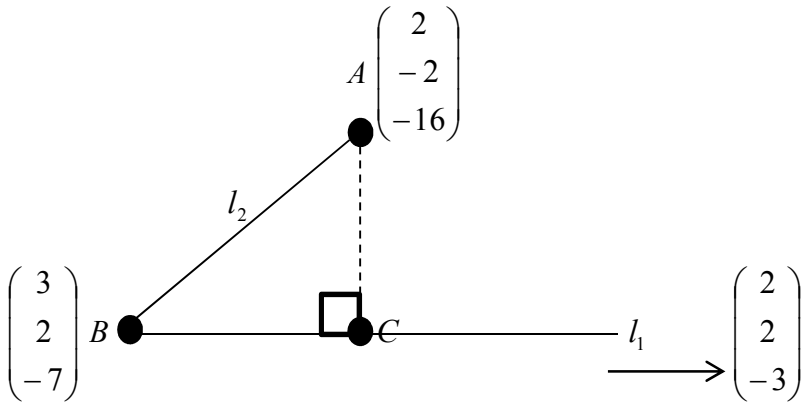
By comparison, $1 + 2\mu = 4 + \lambda \rightarrow \lambda = -3 + 2\mu$

Also, $2\mu = 6 + 4\lambda = 6 + 4(-3 + 2\mu) = -6 + 8\mu$

Solving gives $\mu = 1, \quad \lambda = -1$ (Verify that these values also satisfy $-4 - 3\mu = 2 + 9\lambda$)

$\therefore$ The point of intersection is given by $\begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}$ (shown)

(ii)



Let C be the foot of the perpendicular from point A to the line l_1.

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -9 \end{pmatrix}$$

$$\overrightarrow{BC} = \left[\overrightarrow{BA} \bullet \begin{pmatrix} \hat{2} \\ 2 \\ -3 \end{pmatrix}\right]\begin{pmatrix} \hat{2} \\ 2 \\ -3 \end{pmatrix} = \frac{1}{17}\left[\begin{pmatrix} -1 \\ -4 \\ -9 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}\right]\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$= \frac{1}{17}(-2-8+27)\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

Hence, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -10 \end{pmatrix}$ (shown)

(iii) Let the reflection of the point A (on l_2) in the line l_1 be D (which shall lie on l_3)

Then $\overrightarrow{AD} = 2\overrightarrow{AC} = 2\left[\begin{pmatrix} 5 \\ 4 \\ -10 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix}\right] = 2\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 12 \end{pmatrix}$

The direction vector of the required line of reflection l_3 is given by

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \begin{pmatrix} -1 \\ -4 \\ -9 \end{pmatrix} + \begin{pmatrix} 6 \\ 12 \\ 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$$

$\therefore$ Equation of l_3 is $r = \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} + \gamma\begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$ (shown)

6(i) The equation of line m is $r = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3-\lambda \\ 4+2\lambda \\ 1 \end{pmatrix}$, $\lambda \in \Re$

Since OP is perpendicular to m, $\begin{pmatrix} 3-\lambda \\ 4+2\lambda \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 0$, ie

$-3+\lambda+8+4\lambda = 0 \Rightarrow \lambda = -1$; hence, point P is given by $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$. (shown)

(ii) Assuming that l and m intersect, then $\begin{pmatrix} 6+t \\ a+3t \\ at \end{pmatrix} = \begin{pmatrix} 3-\lambda \\ 4+2\lambda \\ 1 \end{pmatrix}$

$6+t = 3-\lambda \Rightarrow \lambda = -3-t$ ----------(1)

$a+3t = 4+2\lambda$ --------(2)

$at = 1 \Rightarrow a = \frac{1}{t}$ --------(3)

Substituting both (1) and (3) into (2):

$\frac{1}{t} + 3t = 4 + 2(-3-t) = -2-2t$

Multiplying both sides by t,

$1+3t^2 = -2t-2t^2$

$5t^2 + 2t + 1 = 0$

Since discriminant of the immediate above quadratic equation is

$$b^2 - 4ac = (2)^2 - 4(5)(1) = -16 < 0,$$

there will be no real solutions and hence l and m **will not** intersect, ie they **have no common point.** (shown)

(iii) $\left| \begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = \sqrt{10+a^2}\cos 60^\circ = \frac{1}{2}\sqrt{10+a^2}$

$|a| = \frac{1}{2}\sqrt{10+a^2}$

Squaring both sides, $a^2 = \frac{1}{4}\left(10+a^2\right) \Rightarrow 4a^2 = 10+a^2$

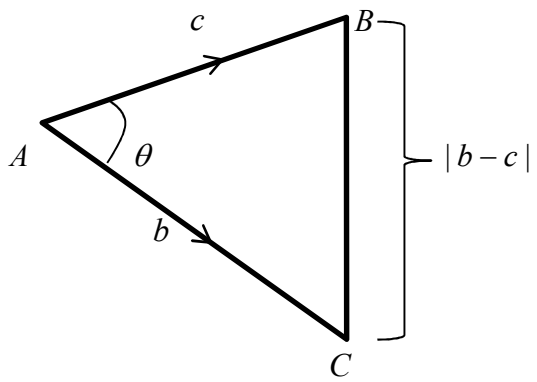
$$3a^2 = 10$$

$$\therefore \quad a = \pm\sqrt{\frac{10}{3}} \text{ (shown)}$$

7. $|b|^2 + |c|^2 - (b-c)\bullet(b-c)$

$= |b|^2 + |c|^2 - [b\bullet b - 2b\bullet c + c\bullet c]$

$= |b|^2 + |c|^2 - [|b|^2 - 2b\bullet c + |c|^2] = 2b\bullet c$



Let the angle between vectors b and c be θ.

Since $(b-c)\bullet(b-c) = |b-c|^2$,

$|b|^2 + |c|^2 - (b-c)\bullet(b-c) = 2b\bullet c$ can be rewritten as

$|b|^2 + |c|^2 - |b-c|^2 = 2b\bullet c$

$|b-c|^2 = |b|^2 + |c|^2 - 2b\bullet c = |b|^2 + |c|^2 - 2|b||c|\cos\theta$ (shown)