Additional Revision Questions

1. The sequence of even integers is grouped in brackets as follows:

{2}, {4,6}, {8, 10, 12}, {14, 16, 18, 20}, {22,……..}

such that each bracket contains one more integer than the preceding bracket.

(a) Show that the total number of integers in the first n brackets is $\frac{n}{2}(n+1)$.

(b) Show also that the sum of all the integers in the first n brackets is

$$\frac{n(n+1)(n^2+n+2)}{4}.$$

(c) Find the first integer in the nth bracket and find also the sum of the integers in the nth bracket.

2. In an infinite series of concentric circles, the radius of the first circle is 12cm, and the radius of each circle after the first is $\frac{2}{3}$ that of the previous circle.

(i) the radius of the n^{th} circle is less than 1 cm. Find the least value of n.

(ii) Find the total area of this infinite series of concentric circles, leaving your answer in terms of π.

3. Prove by induction that $\sum_{r=1}^{n}(r+1)2^r = n(2^{n+1})$.

4. Consider the sequence [u_i, i=1,2,3,……| $u_n = \frac{e-1}{e^n} - \frac{1}{n(n+1)}$], show that $S_N = \sum_{r=1}^{N} u_r = \frac{1}{N+1} - e^{-N}$. Deduce that $S_N > 0$.

5. The functions f and g are defined as follows:

$f(x) \to x^2 + 2x, \quad x \in D$

$g(x) \to |x+4|, \quad x \in \Re$

(i) Determine the set D such that it is the largest possible domain on which f has an inverse and that fg does not exist.

(ii) Find the range of values of x for which $gf(x) > 12$.

6. (a) The function f is defined by

$f(x) \to x^2 - 2x + k, \quad x \in \Re$, where k is a constant.

Show that f is not one-one.

State the largest possible domain of f in the form $(-\infty, \alpha]$, $\alpha \in \Re$, for which the inverse function, f^{-1}, exists.

Find the value of k such that the graphs of f and f^{-1} intersect at the point where $x = -\frac{1}{2}$.

(b) The functions g and h are defined by

$g(x) \to x^2 - 2x, \ x \in \Re$,

$h(x) \to \frac{1}{x}, \quad x \geq 1$.

Define gh and find its range.

7. Prove, by mathematical induction, that $\sum_{r=1}^{n} r(2r-1) = \frac{1}{6}n(n+1)(4n-1)$, where n is a positive integer. By writing down the rth term of the series S=(1)(2)+(3)(4)+(5)(6)+………, deduce the sum of the first n terms of the series S.

8. By using the substitution $x = \tan\theta$, show that

$\int \frac{1}{\sqrt{x^2+1}} dx = \ln|\sec(\tan^{-1} x) + x| + C$, where C is an arbitrary constant.

The region R is bounded by the curves $\frac{1}{\sqrt{x^2+1}}$, the line $y = \frac{1}{2\sqrt{3}}x$ and the positive y-axis.

(i) On a single diagram, sketch the curve of $y = \frac{1}{\sqrt{x^2+1}}$ and the line $y = \frac{1}{2\sqrt{3}}x$, indicating clearly the region R and any intersections with the axes.

(ii) Find the exact area of region R.

(iii) Region S is bounded by the curve $y = \frac{1}{\sqrt{x^2+1}}$, the line y=0.5 and the positive y-axis. This region is rotated through 2π about the line y=0.5. Find the volume of the solid generated.

*9. (a) The sequence $u_1, u_2, u_3, ...$ is defined by $u_n = 5n^2 - 2n$. Show that the **difference** between consecutive terms of the sequence forms an arithmetic progression. Find the sum of the first 50 terms of this progression.

(b) A pump is used to extract air from a bottle. For each operation, the pump can only extract 5% of air in the bottle. The volume of air in the bottle before extraction is $50cm^3$.

(i) Show that the total volume of air extracted after n operations is given by $50(1-0.95^n)$.

(ii) Find the least possible number of operations needed for the pump to extract at least half of the air in the bottle.
Is it possible to extract all the air in the bottle? Justify your answer.

10. The r^{th} term of a sequence is given by

$$u_r = \frac{2r+1}{r^2(r+1)^2}, \quad \text{for r=1,2,3……….}$$

(i) write down the first four terms of the sequence, and hence state the values of $\sum_{r=1}^{n} u_r$ for n=1,2,3 and 4.

(ii) Make a conjecture for a formula for $\sum_{r=1}^{n} u_r$ in terms of n, and prove your formula by mathematical induction.

(iii) Hence find $\sum_{r=3}^{n-1} u_r$ in terms of n.

(iv) State the value of $\sum_{r=1}^{\infty} u_r$.

11. A plane Π_1 has equation $r \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2.$

(i) Find a Cartesian equation of the plane Π_2 which is parallel to the y-z plane and contains the point (-4,1,0).

(ii) A point P (a, 5 ,b) lies on both Π_1 and Π_2. Write down the values of a and b and find the vector equation of l, the line of intersection of planes Π_1 and Π_2.

(iii) Another point Q has coordinates (4,5,6).
Find the coordinates of the foot of the perpendicular from Q to l.
Find the shortest distance from Q to Π_1.
Hence or otherwise, show that $\sin\theta = \frac{\sqrt{5}}{5}$, where θ is the acute angle between the line PQ and Π_1.

12. The plane π_1 has equation $r \bullet$ (-**i** + 2**k**) =-4 and the points A and P have position vectors 4i and **i** + α **j** +**k** respectively, where $\alpha \in \Re$.

(i) Show that A lies on π_1 but P does not.

(ii) Find, in terms of α, the position vector of N, the foot of the perpendicular of P on π_1.

(iii) The plane π_2 contains the points a, P and N, Show that the equation of π_2 is $r \bullet (2\alpha\,\mathbf{i} + 5\mathbf{j} + \alpha\,\mathbf{k}) = 8\alpha$ and write down the equation of l, the line of intersection of π_1 and π_2.

The plane π_3 has equation $r \bullet (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$.

(iv) By considering l, or otherwise, find the value of α for which the three planes intersect in a line.

(v) Given that $\alpha = 2$, find the point of intersection of the three planes.

13. Find all the roots of the equation $w^5 = 1$.

(i) Prove that $e^{i\theta} - 1 = e^{i(\frac{\theta}{2})}(2\sin\frac{\theta}{2})i$.

(ii) hence, show that the roots of $\frac{(z+1)^5}{z^5} = 1$ are $\frac{1}{2}(-1 \pm i\cot\frac{\pi}{5})$ and $\frac{1}{2}(-1 \pm i\cot\frac{2\pi}{5})$.

14. (a) Find the cube roots of $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, giving your answers in exactly the form $re^{i\theta}$. Hence or otherwise, solve the equation $\frac{1}{z^6} - \frac{\sqrt{3}}{z^3} + 1 = 0$, giving your answers exactly in the form $re^{i\theta}$.

(b) The point P represents the variable complex number z, where arg(z-3+4i)=$\frac{\pi}{3}$. The point Q represents the variable complex number w, where |w-3-2i|=k, k>0. If the locus of P is a tangent to the locus of Q, sketch both loci on the same diagram. Hence, show that k=3.

(i) If the complex number w_1 is represented by the point of intersection of P and Q, determine the maximum value of |w- w_1|.

(ii) Determine also the range of values of arg(w-3+4i).

15. (a) Find $\int \frac{1}{9 - 4x^2} dx$

(b) Show that $\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + c$, where c is an arbitrary constant.

(i) Deduce that $\int_0^{\frac{\pi}{3}} \cos^2(2y)dy = \frac{\pi}{6} - \frac{\sqrt{13}}{16}$.

(ii) Find the exact value of $\int_0^{\frac{\pi}{3}} x\cos^2(2x)dx$..

16. Find

(i) $\int \frac{x}{(2x^2+5)^8}dx$, (ii) $\int \frac{e^{\tan x}}{(1+\sin x)(1-\sin x)}dx$, (iii) $\int x^2 e^{3x} dx$.

17. A sequence $\{u_n\}$ is defined recursively as

$u_1 = 1+e$ where e is the Euler's constant

$u_{n+1} = 1+e-\frac{e}{u_n}$, where $n \in \mathbb{Z}^+$

(i) Write down u_2 and u_3 and make a conjecture of u_n in the form of $u_n = \frac{1-A}{1-e^n}$ where A is to be determined in terms of e and n.

(ii) Hence prove your conjecture in (i) by mathematical induction.

(iii) Does u_n exist as $n \to \infty$? If it exists, determine its value.

18. Show that $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+....+n} = \sum_{r=1}^{n} \frac{2}{r(r+1)}$.

Prove, by induction, that

$\frac{1}{2}(\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+....+n}) = 1 - \frac{1}{n+1}$.

Deduce $\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$.

19. The line l whose vector equation is r= 5**i** +2**j** -3**k** + α (7**i** -2**j** -3**k**) passes through the point A with position vector 5**i** +2**j** -3**k**. The plane π whose vector equation is r= -5**i** +2**j** +2**k** + β (7**i** -2**j** -3**k**) + γ (9**i** –**j** +**k**) contains the point B with position vector -5**i** +2**j** +2**k.**

(i) Find the position vector of the point P on the line segment AB such that AP:PB=4: 1.

(ii) The plane π_1 contains the line l and the point P.

Write down a vector equation of the line of intersection of π and π_1.

Find the vector equation of π_1 and the angle between π and π_1.

Hence find the ratio of the perpendicular distances from P to the line l and from P to the plane π.

20. (i) The plane π_1 and the line l_1 have equations

$$r \bullet \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 4 \quad \text{and} \quad r = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \text{ where } \lambda \in \Re \text{, respectively.}$$

Show that l_1 lies on π_1.

(ii) Another plane π_2 contains l_1 and is perpendicular to π_1. Find an equation of π_2 in the form $r \bullet n = p$.

(iii) A third plane π_3 ha a normal parallel to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and passes through the origin.

Write down an equation for π_3.

(iv) A fourth plane π_4 has Cartesian equation x+z=4. Determine whether the 3 planes π_1, π_3 and π_4, intersect. If they do intersect, find the point of line of intersection.

21. (a) Given that $\frac{d^2x}{d\theta} = \sin 7\theta \sin 3\theta$, and $x = 0$, $\frac{dx}{d\theta} = 0$ when $\theta = 0$, find x in terms of θ. Hence find all other values of $\theta \in [-\pi, \pi]$ such that x=0.

(b) The variables x and y are related by

$$xe^y \frac{dy}{dx} + e^y = 2x. \qquad (1)$$

(i) by means of the substitution $z = xe^y$, obtain a differential equation relating z and x.

(ii) Hence, or otherwise, show that the general solution of (1) is $y = \ln(x + \frac{k}{x})$. where k is an arbitrary constant.

(iii) Sketch the solution curve for k=-1, stating clearly any asymptotes and axial intercepts.