Additional Revision Questions Solutions

1(a) In general, n^{th} bracket contains n terms

$\therefore$ sum of number of term in n brackets $= 1+2+3+........+n = \frac{n}{2}(n+1)$ (shown)

(b) Sum of integers in n brackets

$$= \frac{1}{2}\left[\frac{n}{2}(n+1)\right]\left\{2(2)+\left[\frac{n}{2}(n+1)-1\right](2)\right\} = \frac{n}{4}(n+1)[4+n(n+1)-2]$$

$$= \frac{n}{4}(n+1)\left(n^2+n+2\right) \text{ (shown)}$$

(c) Total number of terms in $(n-1)$ brackets $= \frac{n-1}{2}(n-1+1) = \frac{n-1}{2}(n)$

Last term in $(n-1)^{th}$ bracket $= 2+\left[\frac{n-1}{2}(n)-1\right](2) = 2+(n-1)(n)-2 = n^2-n$

$\therefore$ First term in n^{th} bracket $= n^2-n+2$ (shown)

Sum of integers in n^{th} bracket $= \frac{n}{2}\left[2\left(n^2-n+2\right)+(n-1)(2)\right] = (n)(n^2+1)$ (shown)

2 (i) $12\left(\frac{2}{3}\right)^{n-1} < 1 \Rightarrow \left(\frac{2}{3}\right)^{n-1} < \frac{1}{12}$

Solving gives $n > 7.205$, $\therefore$ minimum value of $n = 8$ (shown)

(ii) $$\sum_{i=1}^{\infty} A_i = \pi\left[(12)^2+(12)^2\left(\frac{2}{3}\right)^2+(12)^2\left(\frac{2}{3}\right)^4+.................\right]$$

$$= \pi\left[\frac{12^2}{1-\left(\frac{2}{3}\right)^2}\right] = \frac{1296}{5}\pi \text{ (shown)}$$

3. Let the proposition P_n be $\sum_{r=1}^{n}(r+1)2^r = n\left(2^{n+1}\right)$, where $n \in \mathbb{Z}^+$

Considering P_1: LHS $= (1+1)(2) = 4$ and $(1)(2^2) = 4$; since LHS=RHS, P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, ie $\sum_{r=1}^{k}(r+1)2^r = k\left(2^{k+1}\right)$

For P_{k+1} : $\sum_{r=1}^{k+1}(r+1)2^r = \sum_{r=1}^{k}(r+1)2^r + (k+2)2^{k+1} = k\left(2^{k+1}\right) + (k+2)2^{k+1}$

$$= 2^{k+1}(k+k+2) = 2^{k+1}(2k+2) = 2^{k+2}(k+1)$$

P_k is true $\Rightarrow$ is P_{k+1} true. Since P_1 is true, by Mathematical Induction,

$$\sum_{r=1}^{n}(r+1)2^r = n\left(2^{n+1}\right) \text{ for all } n \in \mathbb{Z}^+ \text{ (shown)}$$

4. $\sum_{r=1}^{N} U_r = \sum_{r=1}^{N} \frac{e-1}{e^r} - \frac{1}{r} + \frac{1}{r+1} =$

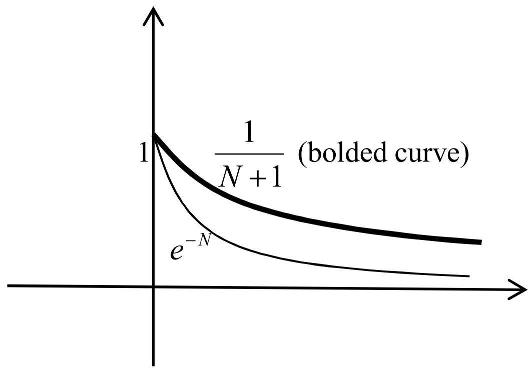
$$\left\{ \begin{array}{l} \frac{e-1}{e^1} \quad -1 + \frac{1}{2} \\ \frac{e-1}{e^2} \quad -\frac{1}{2} + \frac{1}{3} \\ \frac{e-1}{e^3} \quad -\frac{1}{3} + \frac{1}{4} \\ \quad \vdots \\ \frac{e-1}{e^N} \quad -\frac{1}{N} + \frac{1}{N+1} \end{array} \right.$$

$$= (e-1)\left[\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \ldots\ldots\ldots \frac{1}{e^N}\right] - 1 + \frac{1}{N+1}$$

$$= (e-1)\left[\frac{\left(\frac{1}{e}\right)\left(1 - \left(\frac{1}{e}\right)^N\right)}{1 - \frac{1}{e}}\right] - 1 + \frac{1}{N+1}$$

$$= (e-1)\left[\frac{\left(\frac{1}{e}\right)\left(\frac{e^N - 1}{e^N}\right)}{\frac{e-1}{e}}\right] - 1 + \frac{1}{N+1} = \frac{e^N - 1}{e^N} - 1 + \frac{1}{N+1}$$

$$= 1 - e^{-N} - 1 + \frac{1}{N+1} \quad = \frac{1}{N+1} - e^{-N} \text{ (shown)}$$

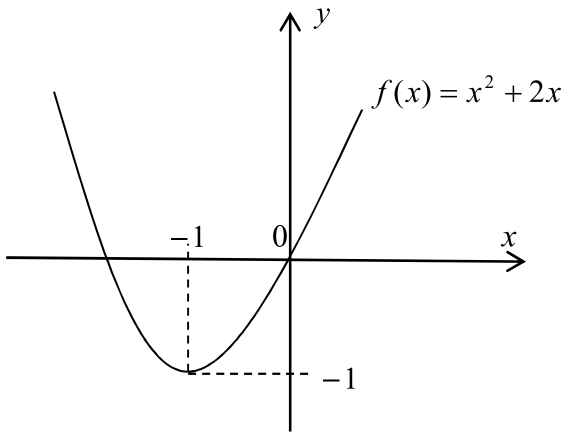


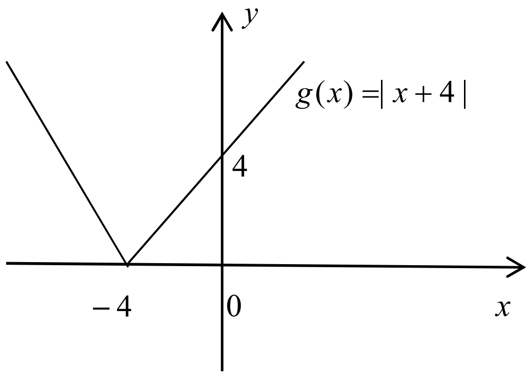
From the graph on the left, it is observed that

$= \frac{1}{N+1} > e^{-N}$ for all $N > 0$

Hence, $S_N > 0$ (shown)

5(i)





$R_g = [0, \infty)$; for f^{-1} to exist, largest possible domains for f are $x \le -1$ or $x \ge -1$.

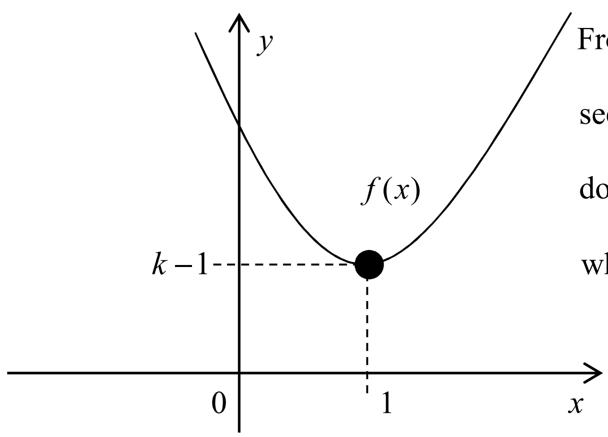
However, since $fg(x)$ DOES NOT exist, ie $R_g \not\subseteq D_f$, $x \ge -1$ is rejected and the largest domain satisfying all conditions is $x \le -1$. (shown)

(ii) $gf(x) = |x^2 + 2x + 4| = (x+2)^2$

For $gf(x) > 12$, $(x+2)^2 > 12 \rightarrow x^2 + 2x - 8 > 0 \rightarrow (x+4)(x-2) > 0$

$\therefore \quad x < -4$ or $x > 2$ (shown)

6(a) $f(2) = f(0) = k$, hence f is not one-one. (shown)



From the graph on the left, it can be seen that the largest possible domain for f^{-1} to exist is $(-\infty, 1]$ where $\alpha = 1$ (shown)

When f and f^{-1} intersect, $f(x) = x \Rightarrow x^2 - 2x + k = x$

Substituting $x = -\frac{1}{2}$ gives $k = 3x - x^2 = 3\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2 = -\frac{7}{4}$ (shown)

(b) $gh(x) = \left(\frac{1}{x}\right)^2 - \frac{2}{x} = \frac{1}{x^2} - \frac{2}{x}, \quad x \geq 1$ (shown)

$D_g \cap R_h = (-\infty, \infty) \cap (0, 1] = (0, 1]$

Substituting this new domain $= (0, 1]$ into g gives $R_{gh} = [-1,\ 0)$ (shown)

7. Let the proposition P_n be $\sum_{r=1}^{n} r(2r-1) = \frac{1}{6}n(n+1)(4n-1),$, where $n \in \mathbb{Z}^+$

Considering P_1 : LHS$=(1)(2-1) = 1$, RHS$=\frac{1}{6}(1)(2)(3) = 1$; since LHS=RHS, P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, ie $\sum_{r=1}^{k} r(2r-1) = \frac{1}{6}k(k+1)(4k-1)$

For P_{k+1} : $\sum_{r=1}^{k+1} r(2r-1) = \sum_{r=1}^{k} r(2r-1) + (k+1)[2(k+1)-1]$

$$= \frac{1}{6}k(k+1)(4k-1) + (k+1)(2k+1) = \frac{1}{6}(k+1)[k(4k-1) + 6(2k+1)]$$

$$= \frac{1}{6}(k+1)(4k^2 + 11k + 6) = \frac{1}{6}(k+1)(k+2)(4k+3)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][4(k+1)-1]$$

P_k is true$\Rightarrow$ is P_{k+1} true. Since, by Mathematical Induction,

$$\sum_{r=1}^{k} r(2r-1) = \frac{1}{6}k(k+1)(4k-1) \text{ for all } k \in \mathbb{Z}^+ \text{ (shown)}$$

r^{th} term of series $= [1 + (r-1)(2)][2 + (r-1)(2)] = (2r-1)(2r)$

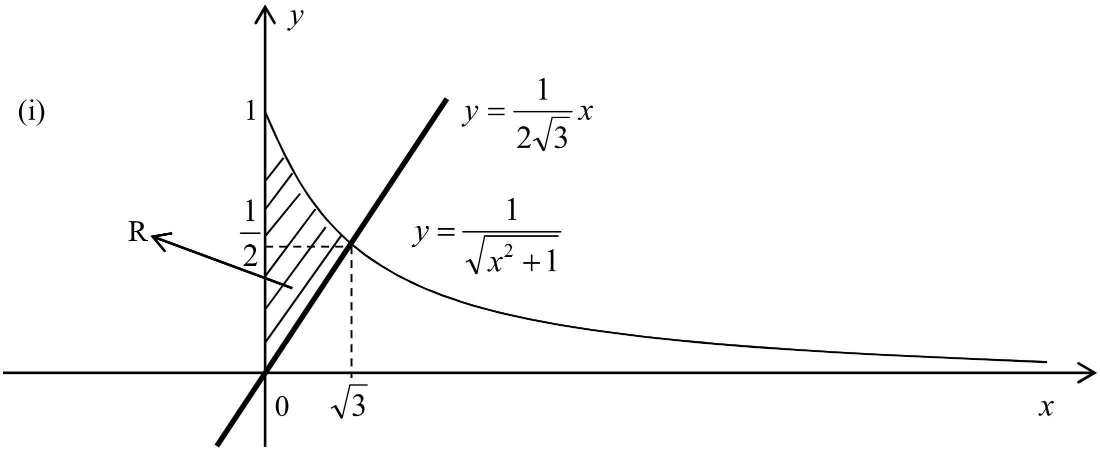
$\therefore (1)(2) + (3)(4) + (5)(6) + \ldots\ldots\ldots\ldots\ldots = \sum_{r=1}^{n}(2r-1)(2r) = 2\sum_{r=1}^{n}(2r-1)(r)$

$= 2\left[\frac{1}{6}n(n+1)(4n-1)\right] = \frac{1}{3}n(n+1)(4n-1)$ (shown)

8. $\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{1+\tan^2\theta}}\left(\sec^2\theta\right)d\theta = \int \sec\theta\ d\theta$

$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\sec\left(\tan^{-1}x\right) + x\right| + C$ (shown)

(i)



(ii) Area of R $= \int_0^{\sqrt{3}} \frac{1}{\sqrt{x^2+1}}\,dx - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \left[\ln\left|\sec\left(\tan^{-1}x\right) + x\right|\right]_0^{\sqrt{3}} - \frac{\sqrt{3}}{4}$

$= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{4}$ sq units (shown)

(iii) $V = \pi \int_0^{\sqrt{3}} \left(y - \frac{1}{2}\right)^2 dx = \int_0^{\sqrt{3}} y^2 - y + \frac{1}{4}\ dx = \pi \int_0^{\sqrt{3}} \frac{1}{x^2+1} - \frac{1}{\sqrt{x^2+1}} + \frac{1}{4}\,dx$

$= \pi\left[\tan^{-1}x - \ln\left|\sec\left(\tan^{-1}x\right) + x\right| + \frac{1}{4}x\right]_0^{\sqrt{3}} = \pi\left[\frac{\pi}{3} - \ln(2+\sqrt{3}) + \frac{\sqrt{3}}{4}\right]$

$= 0.513$ cubic units (shown)

9 (a) $T_n = u_{n+1} - u_n = 5(n+1)^2 - 2(n+1) - \left(5n^2 - 2n\right) = 10n + 3$

$T_n - T_{n-1} = (10n+3) - [10(n-1)+3] = 10$

$\therefore$ The difference between consecutive terms of the sequence forms an AP. (shown)

$S_{50} = \frac{50}{2}\left[2(13) + (50-1)(10)\right] = 12900$ (shown)

(b) (i) After 1st operation, volume extracted $= 0.05(50)$

volume left $= 50 - 0.05(50) = 0.95(50)$

After 2nd operation, volume extracted $= 0.05(0.95)(50)$

total volume extracted $= 0.05(50) + 0.05(0.95)(50)$

$$= 0.05(50)[1 + 0.95] = \frac{0.05(50)\left(1 - 0.95^2\right)}{1 - 0.95}$$

$$= 50\left(1 - 0.95^2\right)$$

volume left $= 50 - 50\left(1 - 0.95^2\right) = 50\left(0.95^2\right)$

After 3[rd] operation, volume extracted $= 0.05(50)\left(0.95^2\right)$

total volume extracted $= 0.05(50) + 0.05(0.95)(50) + 0.05(50)\left(0.95^2\right)$

$$= 0.05(50)[1 + 0.95 + 0.95^2] = \frac{0.05(50)\left(1 - 0.95^3\right)}{1 - 0.95}$$

$$= 50\left(1 - 0.95^3\right)$$

Hence, after n operations, total volume extracted $= 50\left(1 - 0.95^n\right)$ (shown)

(ii) Total volume $> 50(0.5) = 25 \Rightarrow 50\left(1 - 0.95^n\right) > 25$

$n > 13.5 \rightarrow$ least value of $n = 14$ (shown)

It is **not** possible to extract all the air in the bottle unless an infinite (unattainable) number of operations have been carried out. (shown)

10 (i) $u_1 = \frac{3}{4}$, $u_2 = \frac{5}{36}$, $u_3 = \frac{7}{144}$, $u_4 = \frac{9}{400}$

$\sum_{r=1}^{n} u_r = \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}$ for $n = 1,\ 2,\ 3$ and 4 respectively (shown)

(ii) $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2} =$

$$\left\{\begin{array}{l} 1 - \frac{1}{4} \\ \frac{1}{4} - \frac{1}{9} \\ \frac{1}{9} - \frac{1}{16} \\ \quad \vdots \\ \frac{1}{n^2} - \frac{1}{(n+1)^2} \end{array}\right.$$

$= 1 - \frac{1}{(n+1)^2}$ (shown)

(iii) Let the proposition P_n be $\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}$, where $n\in\mathbb{Z}^+$

Considering P_1: LHS=$\frac{2+1}{(1+1)^2}=\frac{3}{4}$, RHS=$1-\frac{1}{4}=\frac{3}{4}$; since LHS=RHS, P_1 is true.

Assume P_k is true for some $k\in\mathbb{Z}^+$, ie $\sum_{r=1}^{k}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(k+1)^2}$

For P_{k+1}: $\sum_{r=1}^{k+1}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=1}^{k}\frac{2r+1}{r^2(r+1)^2}+\frac{2(k+1)+1}{(k+1)^2(k+2)^2}$

$$=\frac{2(k+1)+1}{(k+1)^2(k+2)^2}+1-\frac{1}{(k+1)^2}$$

$$=1-\frac{1}{(k+1)^2}\left[1-\frac{2k+3}{(k+2)^2}\right]$$

$$=1-\frac{1}{(k+1)^2}\left[\frac{(k+2)^2-2k-3}{(k+2)^2}\right]=1-\frac{1}{(k+1)^2}\left[\frac{(k+1)^2}{(k+2)^2}\right]$$

$$=1-\frac{1}{(k+2)^2}$$

P_k is true $\Rightarrow$ is P_{k+1} true. Since P_1 is true, by Mathematical Induction,

$$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}\text{ for all } n\in\mathbb{Z}^+ \text{ (shown)}$$

$\sum_{r=3}^{n-1}u_r=\sum_{r=1}^{n-1}u_r-\sum_{r=1}^{2}u_r=1-\frac{1}{(n-1+1)^2}-\frac{8}{9}=\frac{1}{9}-\frac{1}{n^2}$ (shown)

(iv) $\sum_{r=1}^{\infty 1}u_r=1$ (shown)

11(i) Vector normal to $\prod_2$ is $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\therefore\ r\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-4\\1\\0\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-4$ and Cartesian equation of $\prod_2$ is $x=-4$ (shown)

(ii) For $\prod_1$, $\begin{pmatrix} a \\ 5 \\ b \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2 \Rightarrow b = 2$; for $\prod_2$, $\begin{pmatrix} a \\ 5 \\ b \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -4 \Rightarrow a = -4$ (shown)

Direction vector of line of intersection $= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

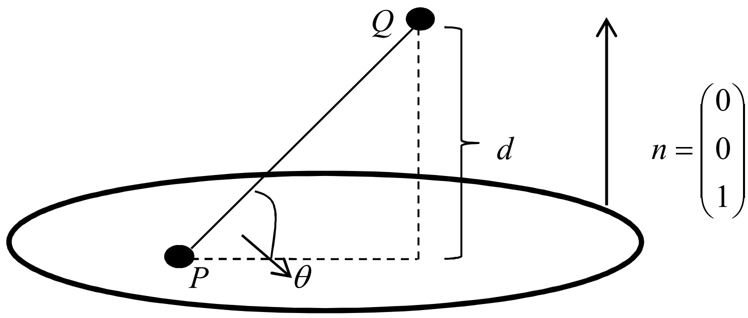
$\therefore$ Equation of line of intersection is $r = \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (shown)

(iii) $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}$

Since $\overrightarrow{PQ} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$, point Q is directly above point $P \Rightarrow$ foot of perpendicular from Q

to line $l = \overrightarrow{OP} = \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix}$ (shown)

(iv)



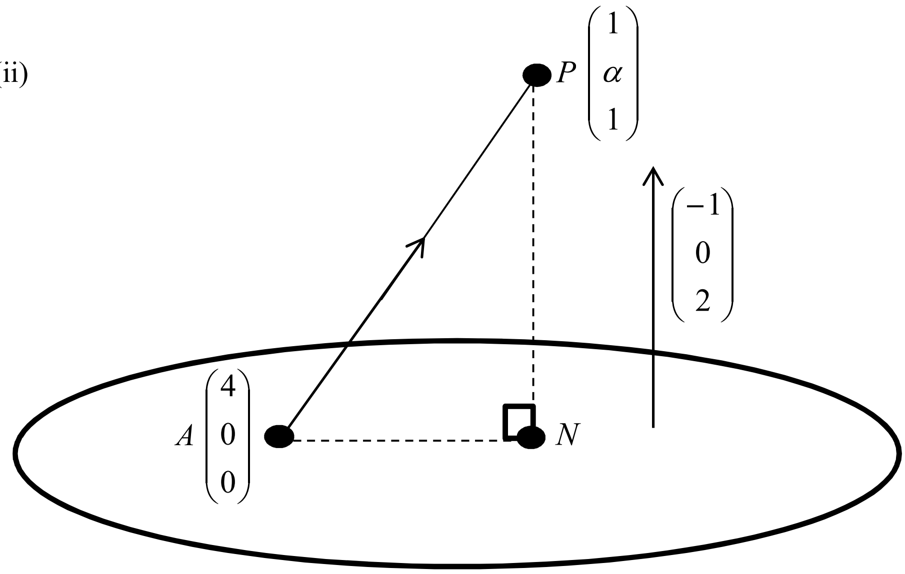
Shortest distance from Q to $\prod_1 = \left| \overrightarrow{PQ} \bullet \hat{n} \right| = 4$ units(shown)

$\therefore \sin\theta = \dfrac{4}{\left| \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \right|} = \dfrac{4}{\sqrt{80}} = \dfrac{1}{\sqrt{5}}$ (shown)

12(i) For point A, $= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -4$; For point P, $= \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 1 \neq -4$

$\therefore$ Point A lies on π_1 but point P does not. (shown)

(ii)



$$\overrightarrow{NP} = \left[\overrightarrow{AP} \bullet \hat{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}\right] \hat{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}} = \left[\begin{pmatrix} -3 \\ \alpha \\ 1 \end{pmatrix} \bullet \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\right] \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} \Rightarrow \overrightarrow{ON} = \overrightarrow{OP} - \overrightarrow{NP} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \alpha \\ -1 \end{pmatrix} \text{ (shown)}$$

(iii) For π_2, $n_2 = \overrightarrow{AP} \times \overrightarrow{AN} = \begin{pmatrix} -3 \\ \alpha \\ 1 \end{pmatrix} \times \left[\begin{pmatrix} 2 \\ \alpha \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} -2\alpha \\ -5 \\ -\alpha \end{pmatrix} // \begin{pmatrix} 2\alpha \\ 5 \\ \alpha \end{pmatrix}$

$\therefore$ Equation of π_2 is $r \bullet \begin{pmatrix} 2\alpha \\ 5 \\ \alpha \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2\alpha \\ 5 \\ \alpha \end{pmatrix} = 8\alpha$ (shown)

Line of intersection of π_1 and π_2 has the direction vector $\overrightarrow{AN} = \begin{pmatrix} -2 \\ \alpha \\ -1 \end{pmatrix}$

$\therefore$ Equation of line of intersection is $r = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ \alpha \\ -1 \end{pmatrix}$ (shown)

(iv) If all 3 planes intersect in a line, then for π_3, direction vector of l (ie line of intersection of π_1 and π_2) must be perpendicular to n_3.

Hence, $\begin{pmatrix} -2 \\ \alpha \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow \alpha = 4$ (shown)

(v) The augmented matrix when $\alpha = 2$ is given by $\left(\begin{array}{ccc|c} -1 & 0 & 2 & -4 \\ 4 & 5 & 2 & 16 \\ 1 & 1 & 2 & 4 \end{array}\right)$

Solving this gives the common point of intersection $= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ (shown)

13. $w^5 = 1 = e^{(i)(2k\pi)} \Rightarrow w = e^{(i)\left(\frac{2k\pi}{5}\right)}, \quad k = 0, \ \pm 1, \ \pm 2$

$\therefore \quad w = 1, e^{(i)\left(\frac{2\pi}{5}\right)}, e^{(i)\left(-\frac{2\pi}{5}\right)}, e^{(i)\left(\frac{4\pi}{5}\right)}, e^{(i)\left(-\frac{4\pi}{5}\right)}$ (shown)

(i) $e^{i\theta} - 1 = e^{(i)\left(\frac{\theta}{2}\right)}\left[e^{(i)\left(\frac{\theta}{2}\right)} - e^{(i)\left(-\frac{\theta}{2}\right)}\right] = e^{(i)\left(\frac{\theta}{2}\right)}\left(2\sin\frac{\theta}{2}\right)(i)$ (shown)

(ii) Let $w = \frac{z+1}{z} \to z = \frac{1}{w-1} = \frac{1}{e^{i\theta} - 1} = \frac{1}{e^{(i)\left(\frac{\theta}{2}\right)}\left(2\sin\frac{\theta}{2}\right)(i)} = -\frac{1}{2} ie^{(i)\left(-\frac{\theta}{2}\right)} \cos ec\frac{\theta}{2}$

$= -\frac{1}{2}(i)\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]\cos ec\frac{\theta}{2} = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$

$= \frac{1}{2}\left(-1 - i\cot\frac{\theta}{2}\right)$ where $\theta = \pm\frac{2\pi}{5}, \ \pm\frac{4\pi}{5}$

$\therefore$ The roots are given by $\frac{1}{2}\left(-1 \pm i\cot\frac{\pi}{5}\right)$ and $\frac{1}{2}\left(-1 \pm i\cot\frac{2\pi}{5}\right)$ (shown)

14. (a) Let w be the root(s) of the equation $w^3 = \frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{(i)\left(2k\pi + \frac{\pi}{6}\right)}$,

ie $w = e^{(i)\left(\frac{2k\pi}{3} + \frac{\pi}{18}\right)}$ where $k = 0,\ 1, 2$

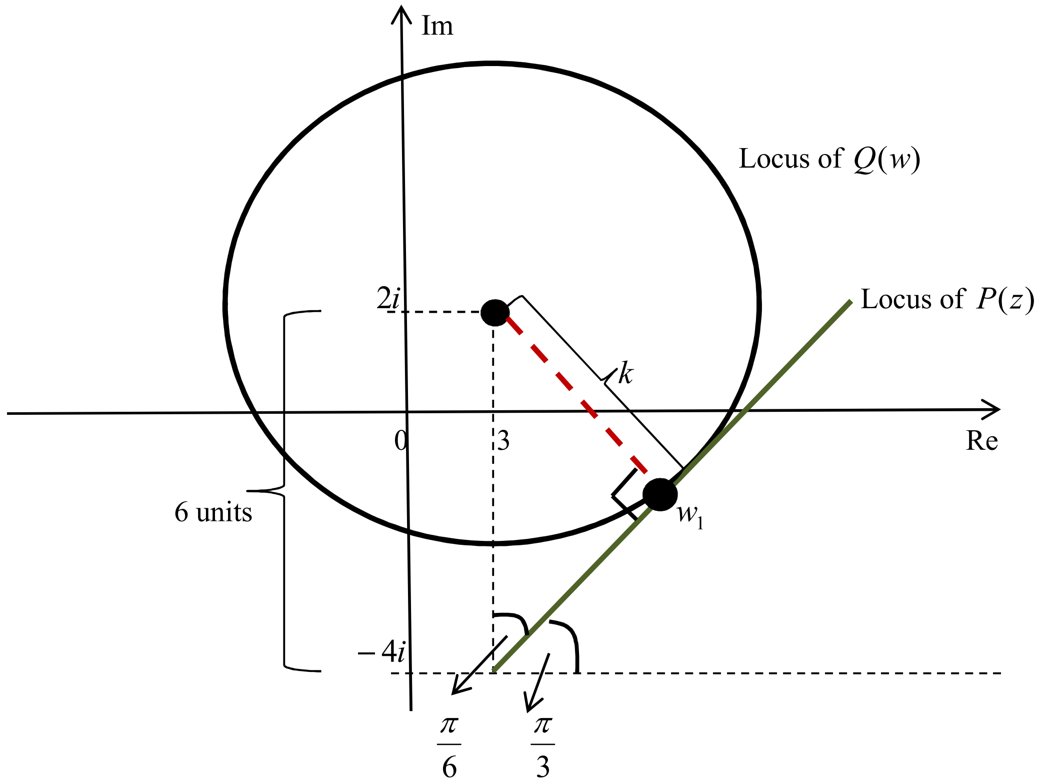
Hence, $w = e^{(i)\left(\frac{\pi}{18}\right)}, e^{(i)\left(-\frac{11\pi}{18}\right)}, e^{(i)\left(\frac{13\pi}{18}\right)}$ (shown)

$\frac{1}{z^6} - \frac{\sqrt{3}}{z^3} + 1 = 0 \Rightarrow 1 - \sqrt{3}z^3 + z^6 = 0$

$\therefore\ z^3 = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}i$

and $z = e^{(i)\left(\pm\frac{\pi}{18}\right)}, e^{(i)\left(\pm\frac{11\pi}{18}\right)}, e^{(i)\left(\pm\frac{13\pi}{18}\right)}$ (shown)

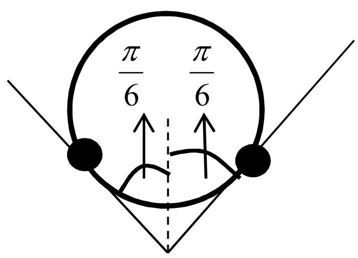
(b)



From the above Argand Diagram, observing the right angle triangle, $\sin\frac{\pi}{6} = \frac{k}{6} \Rightarrow k = 3$ (shown)

(i) Maximum value of $|w - w_1| = 6$ (shown)

(ii)



Range is $\frac{\pi}{3} \le \arg(w - 3 + 4i) \le \frac{2\pi}{3}$ (shown)

15(a) $\int \frac{1}{9-4x^2}dx = \frac{1}{2}\int \frac{2}{3^2-(2x)^2}dx = \frac{1}{2}\left(\frac{1}{4}\right)\ln\left|\frac{3+2x}{3-2x}\right| + C = \frac{1}{8}\ln\left|\frac{3+2x}{3-2x}\right| + C$ (shown)

(b) $\int \cos^2 x \ dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$ (shown)

(i) Let $2y = x$, then $\int_0^{\frac{\pi}{3}} \cos^2(2y)dy = \frac{1}{2}\int_0^{\frac{2\pi}{3}} \cos^2 x \ dx$

$$= \frac{1}{2}\left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{2\pi}{3}} = \frac{\pi}{6} - \frac{\sqrt{3}}{16} \text{ (shown)}$$

(ii) $\int_0^{\frac{\pi}{3}} x\cos^2(x)dx = \left[\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)(x)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \frac{1}{2}x + \frac{1}{4}\sin 2x \ dx$

$$= \frac{\pi}{3}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - \left[\frac{1}{4}x^2 - \frac{1}{8}\cos 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi^2}{18} + \frac{\pi}{8\sqrt{3}} - \left[\left(\frac{\pi^2}{36} + \frac{1}{16}\right) + \frac{1}{8}\right] = \frac{\pi^2}{36} + \frac{\pi}{8\sqrt{3}} - \frac{3}{16} \text{ (shown)}$$

16(i) $\int \frac{x}{(2x^2+5)^8}dx = \frac{1}{4}\int (x)(2x^2+5)^{-8}dx = \frac{1}{4}\frac{(2x^2+5)^{-7}}{(-7)} + C = -\frac{1}{28(2x^2+5)^7} + C$ (shown)

(ii) $\int \frac{e^{\tan x}}{(1+\sin x)(1-\sin x)} dx = \int \frac{e^{\tan x}}{1-\sin^2 x}dx = \int \frac{e^{\tan x}}{\cos^2 x} dx$

$$= \int \sec^2 x\left(e^{\tan x}\right)dx = e^{\tan x} + C \text{ (shown)}$$

(iii) $\int x^2 e^{3x} dx = \left(\frac{1}{3}e^{3x}\right)(x^2) - \int (2x)\left(\frac{1}{3}e^{3x}\right)dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x}dx$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx\right] = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C \text{ (shown)}$$

17 (i) $u_2 = 1 + e - \frac{e}{1+e} = \frac{(1+e)^2 - e}{1+e} = \frac{e^2+e+1}{1+e} = \frac{(e^2+e+1)(1-e)}{(1+e)(1-e)} = \frac{1-e^3}{1-e^2}$

$u_3 = 1 + e - e\left(\frac{1-e^2}{1-e^3}\right) = \frac{1-e^4}{1-e^3}$ $\quad \therefore$ Conjecture is $u_n = \frac{1-e^{n+1}}{1-e^n}$ (shown)

(ii) Let the proposition P_n be $u_n = \dfrac{1-e^{n+1}}{1-e^n}$, where $n \in \mathbb{Z}^+$

Considering P_1: $u_1 = \dfrac{1-e^2}{1-e} = \dfrac{(1+e)(1-e)}{1-e} = 1+e$, hence P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, ie $u_k = \dfrac{1-e^{k+1}}{1-e^k}$

For P_{k+1} : $u_{k+1} = 1+e-\dfrac{e}{u_k} = 1+e-e\left(\dfrac{1-e^k}{1-e^{k+1}}\right)$

$$= \frac{\left(1-e^{k+1}\right)\left(1+e\right)-e+e^{k+1}}{1-e^{k+1}} = \frac{1-e^{k+2}}{1-e^{k+1}}$$

P_k is true $\Rightarrow$ is P_{k+1} true. Since P_1 is true, by Mathematical Induction,

$u_n = \dfrac{1-e^{n+1}}{1-e^n}$ for all $n \in \mathbb{Z}^+$ (shown)

(iii) When $n \to \infty$, $u_n = \dfrac{1-e^{n+1}}{1-e^n} \approx \dfrac{-e^{n+1}}{-e^n} = e$ (shown)

18. $\dfrac{1}{1}+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+.........+\dfrac{1}{1+2+.........+n} = \sum_{r=1}^{n}\dfrac{1}{\frac{r}{2}(r+1)} = \sum_{r=1}^{n}\dfrac{2}{r(r+1)}$ (shown)

Let the proposition P_n be

$$\frac{1}{2}\left(\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+.........+\frac{1}{1+2+.........+n}\right) = 1-\frac{1}{n+1},$$

ie $\dfrac{1}{2}\sum_{r=1}^{n}\dfrac{2}{r(r+1)} = \sum_{r=1}^{n}\dfrac{1}{r(r+1)} = 1-\dfrac{1}{n+1}$ where $n \in \mathbb{Z}^+$

Considering P_1 : LHS$=\dfrac{1}{(1)(1+1)} = \dfrac{1}{2}$, RHS$=1-\dfrac{1}{1+1} = \dfrac{1}{2}$; since LHS=RHS, P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, ie $\sum_{r=1}^{k}\dfrac{1}{r(r+1)} = 1-\dfrac{1}{k+1}$

For P_{k+1} : $\sum_{r=1}^{k+1}\dfrac{1}{r(r+1)} = \sum_{r=1}^{k}\dfrac{1}{r(r+1)} + \dfrac{1}{(k+1)(k+2)} = 1-\dfrac{1}{k+1}+\dfrac{1}{(k+1)(k+2)}$

$$= 1-\frac{1}{k+1}\left(1-\frac{1}{k+2}\right) = 1-\frac{1}{k+1}\left(\frac{k+1}{k+2}\right) = 1-\frac{1}{k+2}$$

P_k is true $\Rightarrow$ is P_{k+1} true. Since P_1 is true, by Mathematical Induction,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} \text{ for all } n \in \mathbb{Z}^+ \text{ (shown)}$$

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1 \text{ (shown)}$$

19 (i)

$$\overrightarrow{OP} = \frac{4\overrightarrow{OB} + \overrightarrow{OA}}{5} = \frac{1}{5}\left[4\begin{pmatrix}-5\\2\\2\end{pmatrix} + \begin{pmatrix}5\\2\\-3\end{pmatrix}\right] = \begin{pmatrix}-3\\2\\1\end{pmatrix} \text{ (shown)}$$

(ii) Both π_1 and π contain the direction vector $\begin{pmatrix}7\\-2\\-3\end{pmatrix}$ as well as points A and B.

[Since π_1 contains the line l, then it also contains the point A; since P also lies on π_1,

in addition the chords AP and PB must exist as a continuous line, therefore B also lies on π_1.]

Hence, vector equation of line of intersection is $r = \begin{pmatrix}-5\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}7\\-2\\-3\end{pmatrix}$ (shown)

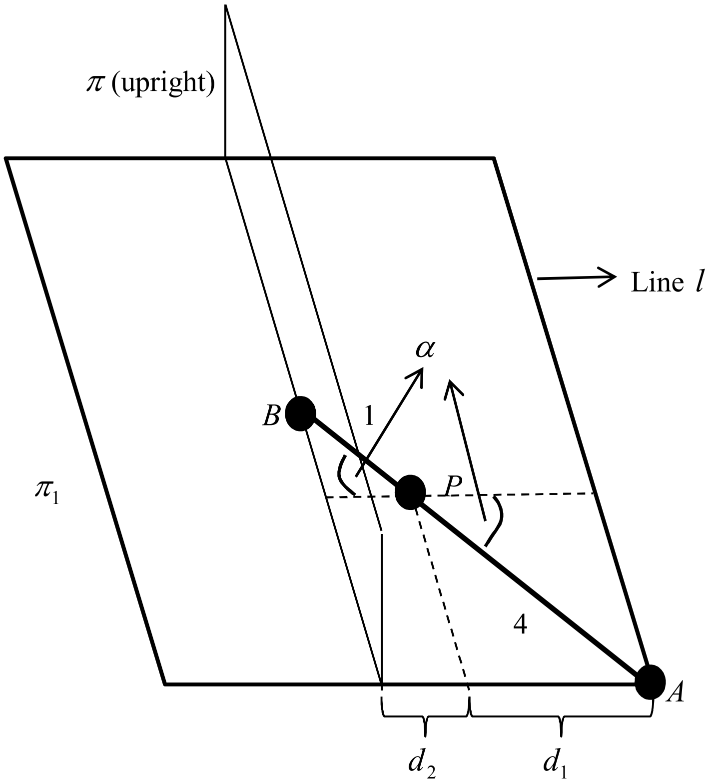
Another direction vector lying on π_1 = $\overrightarrow{AP} = \begin{pmatrix}-3\\2\\1\end{pmatrix} - \begin{pmatrix}5\\2\\-3\end{pmatrix} = \begin{pmatrix}-8\\0\\4\end{pmatrix} // \begin{pmatrix}-2\\0\\1\end{pmatrix}$

$\therefore\ n_1 = \begin{pmatrix}-2\\0\\1\end{pmatrix} \times \begin{pmatrix}7\\-2\\-3\end{pmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix}$ and equation of π_1 : $r \bullet \begin{pmatrix}2\\1\\4\end{pmatrix} = \begin{pmatrix}-3\\2\\1\end{pmatrix} \bullet \begin{pmatrix}2\\1\\4\end{pmatrix} = 0$ (shown)

Let angle between planes π_1 and π be θ

Then $n \bullet n_1 = |n|\,|n_1|\cos\theta \Rightarrow \begin{pmatrix}-5\\-34\\11\end{pmatrix} \bullet \begin{pmatrix}2\\1\\4\end{pmatrix} = \left|\begin{pmatrix}-5\\-34\\11\end{pmatrix}\right| \left|\begin{pmatrix}2\\1\\4\end{pmatrix}\right| \cos\theta$

Solving $\cos\theta = 0$, ie $\theta = 90^\circ$ (shown)



Required ratio $= d_1 : d_2 = 4\cos\alpha : \cos\alpha = 4:1$ (shown)

20 (i) $\begin{pmatrix} \lambda \\ 2+2\lambda \\ 1-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 4 \quad \therefore \quad l_1$ lies on π_1 (shown)

(ii) $n_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 3 \end{pmatrix} // \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

and equation of π_2 : $r \bullet \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = 5$ (shown)

(iii) Equation of π_3 : $r \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ (shown)

(iv) Equation of π_4 : $r \bullet \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4$

Augmented matrix is given by $\left(\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 4 \end{array}\right)$ which is reduced to $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $z = \lambda$, then line of intersection is given by

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ (shown)}$$

21 (a) Using the identity $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, we have

$$\frac{d^2x}{d\theta^2} = \sin 7\theta \sin 3\theta = -\frac{1}{2}(\cos 10\theta - \cos 4\theta)$$

Integrating both sides *wrt* θ,

$$\frac{dx}{d\theta} = -\frac{1}{20}\sin 10\theta + \frac{1}{8}\sin 4\theta + C$$

When $\theta = 0$, $\frac{dx}{d\theta} = 0 \Rightarrow C = 0$

Integrating both sides once again *wrt* θ,

$$x = \frac{1}{200}\cos 10\theta - \frac{1}{32}\cos 4\theta + B$$

When $\theta = 0$, $x = 0 \Rightarrow B = \frac{21}{800}$; hence $x = \frac{1}{200}\cos 10\theta - \frac{1}{32}\cos 4\theta + \frac{21}{800}$ (shown)

When $x = 0$, $\theta = \pm 1.42, \ \pm 1.72, \ \pm 3.14$ (shown)

(b) (i) Differentiating both sides of $z = xe^y$ *wrt* x gives

$\frac{dz}{dx} = e^y + \frac{dy}{dx}(e^y)(x)$; substituting into (1) as defined in the question gives $\frac{dz}{dx} = 2x$ (shown)

(ii) Integrating both sides of $\frac{dz}{dx} = 2x$ *wrt* $x: z = x^2 + k \rightarrow xe^y = x^2 + k$

$$e^y = x + \frac{k}{x} \qquad \therefore \ y = \ln\left(x + \frac{k}{x}\right) \text{ (shown)}$$

(iii)