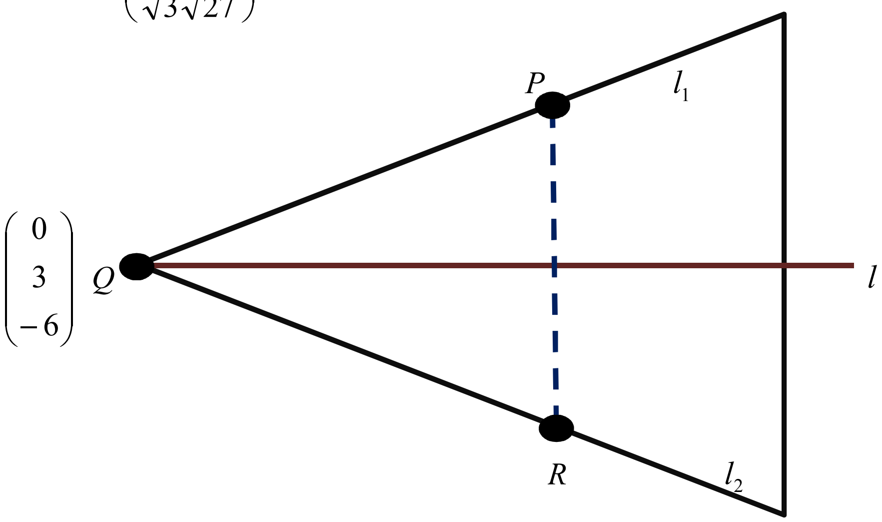
Additional Revision Questions 2 Solutions

1(i) Let the acute angle between l_1 and l_2 be θ. Then

$$\left|\begin{pmatrix}1\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\5\end{pmatrix}\right| = \left|\begin{pmatrix}1\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\5\end{pmatrix}\right|\cos\theta \Rightarrow\ 5=\sqrt{3}\sqrt{27}\cos\theta$$

$$\therefore\ \theta=\cos^{-1}\left(\frac{5}{\sqrt{3}\sqrt{27}}\right)=56.25^\circ \text{ (shown)}$$

(ii)



Recognising that without the modulus function in (i), the angle between l_1 and l_2 would be obtuse, λ must be positive and μ negative (or vice-versa).

Hence, we set $\overrightarrow{QP}=5\hat{m}_1=5\widehat{\begin{pmatrix}1\\1\\-1\end{pmatrix}}=\frac{5}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

and $\overrightarrow{QR}=5\hat{m}_2=-5\widehat{\begin{pmatrix}1\\-1\\5\end{pmatrix}}=\frac{-5}{\sqrt{27}}\begin{pmatrix}1\\-1\\5\end{pmatrix}$ (shown)

(Note: we can also let $\overrightarrow{QP}=-\frac{5}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\overrightarrow{QP}=\frac{5}{\sqrt{27}}\begin{pmatrix}1\\-1\\5\end{pmatrix}$)

Direction vector of line $l=\frac{5}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}+\left[\frac{-5}{\sqrt{27}}\begin{pmatrix}1\\-1\\5\end{pmatrix}\right]=\frac{15}{\sqrt{27}}\begin{pmatrix}1\\1\\-1\end{pmatrix}-\frac{5}{\sqrt{27}}\begin{pmatrix}1\\-1\\5\end{pmatrix}$

$$= \frac{5}{\sqrt{27}}\left[\begin{pmatrix}3\\3\\-3\end{pmatrix}-\begin{pmatrix}1\\-1\\5\end{pmatrix}\right]=\frac{5}{\sqrt{27}}\begin{pmatrix}2\\4\\-8\end{pmatrix} // \begin{pmatrix}1\\2\\-4\end{pmatrix}$$

(Note: the direction vector of the acute angular bisector of two lines is simply given by the vector addition of the direction vectors of these two lines.)

$\therefore$ Equation of line l is given by $r = \begin{pmatrix}0\\3\\-6\end{pmatrix} + \alpha\begin{pmatrix}1\\2\\-4\end{pmatrix}$ (shown)

(iii) Normal to $\prod_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix} \times \begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}4\\-6\\-2\end{pmatrix} // \begin{pmatrix}2\\-3\\-1\end{pmatrix}$

Equation of plane $\prod_1$ is given by $r \bullet \begin{pmatrix}2\\-3\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\-6\end{pmatrix} \bullet \begin{pmatrix}2\\-3\\-1\end{pmatrix} = -3,$

ie $r \bullet \begin{pmatrix}-2\\3\\1\end{pmatrix} = 3$ (shown)

(iv) $\begin{pmatrix}2+\mu\\1-\mu\\4+5\mu\end{pmatrix} \bullet \begin{pmatrix}16\\11\\-1\end{pmatrix} = 32+16\mu+11-11\mu-4-5\mu = 39$

Hence, l_2 lies in $\prod_2$. (shown)

(v) Clearly, the common line of intersection of $\prod_1$ and $\prod_2$ is l_2.

Assuming a certain value of μ exists l_2 for such that all 3 planes intersect at a common point,

ie $\begin{pmatrix}2+\mu\\1-\mu\\4+5\mu\end{pmatrix} \bullet \begin{pmatrix}a\\4\\1\end{pmatrix} = b$

$2a + \mu a + 4 - 4\mu + 4 + 5\mu = b$

$\mu(a+1) = b - 2a - 8 \Rightarrow \mu = \dfrac{b-2a-8}{a+1}$

Hence, if no common point of intersection exists, then $a=-1$ and $b\in\Re$. (shown)

2(a) Another root would be $2+3i$ (shown).

$$(3z-c)[z-(2+3i)][z-(2-3i)]=(3z-c)(z^2-4z+13)$$

$$=3z^3+(-c-12)z^2+(39+4c)-13c$$

Comparing with $3z^3+az^2+43z+b$, we have

$39+4c=43\Rightarrow c=1,\quad a=-c-12=-13$ and $b=-13c=-13$.

$3z^3+az^2+43z+b=0$

$3z^3-13z^2+43z-13=0$

$(3z-1)[z-(2+3i)][z-(2-3i)]=0$

$\therefore$ The third root is given by $z=\frac{1}{3}$. (shown)

The relationship between the roots of the equation $3z^3+az^2+43z+b=0$ and

$3iw^3-aw^2-43iw+b=0$ is given by $z=-iw\Rightarrow w=-\frac{z}{i}=iz$. Hence, the new roots can

be obtained by rotating all roots of the original equation $3z^3+az^2+43z+b=0$ **by 90 degrees anti-clockwise about the origin.** (shown)

(b) $\sqrt{2}(z-1)^4=-1-i=\sqrt{2}e^{(i)\left(2k\pi-\frac{3\pi}{4}\right)}$

$$(z-1)^4=e^{(i)\left(2k\pi-\frac{3\pi}{4}\right)}$$

$$z-1=e^{\frac{i}{4}\left(2k\pi-\frac{3\pi}{4}\right)}\Rightarrow z=1+e^{\frac{i\pi}{16}(8k-3)}$$ where $\alpha=8$ for $k=\pm1,\ 0,\ 2$ (shown)

$$z=1+e^{\frac{i\pi}{16}(8k-3)}=1+\cos\left[\frac{(8k-3)\pi}{16}\right]+i\sin\left[\frac{(8k-3)\pi}{16}\right]$$

$$|z| = \sqrt{\left(1+\cos\left[\frac{(8k-3)\pi}{16}\right]\right)^2 + \sin^2\left[\frac{(8k-3)\pi}{16}\right]}$$

$$= \sqrt{1+2\cos\left[\frac{(8k-3)\pi}{16}\right] + \cos^2\left[\frac{(8k-3)\pi}{16}\right] + \sin^2\left[\frac{(8k-3)\pi}{16}\right]}$$

$$= \sqrt{1+2\cos\left[\frac{(8k-3)\pi}{16}\right]+1} = \sqrt{2+2\cos\left[\frac{(8k-3)\pi}{16}\right]}$$

$$= \sqrt{2+2\left\{2\cos^2\left[\frac{(8k-3)\pi}{32}\right]-1\right\}} = \sqrt{4\cos^2\left[\frac{(8k-3)\pi}{32}\right]} = 2\cos\left[\frac{(8k-3)\pi}{32}\right]$$

[$\because \ \cos 2\theta = 2\cos^2\theta - 1$]

Hence, when $k=1$, $|z| = 2\cos\left[\frac{(8-3)\pi}{16}\right] = 2\cos\frac{5\pi}{32}$ (shown)

3(a) $\int 3^{\sqrt{2x+1}}\,dx = \int\left(3^{\sqrt{2x+1}} \bullet \frac{\ln 3}{\sqrt{2x+1}}\right)\frac{\sqrt{2x+1}}{\ln 3}\,dx$ $\qquad$ [$\because \frac{d}{dx}3^{\sqrt{2x+1}} = 3^{\sqrt{2x+1}} \bullet \frac{\ln 3}{\sqrt{2x+1}}$]

$$= 3^{\sqrt{2x+1}}\left(\frac{\sqrt{2x+1}}{\ln 3}\right) - \int \frac{3^{\sqrt{2x+1}}}{\ln 3}\left(\frac{1}{2}\right)(2)(2x+1)^{-\frac{1}{2}}\,dx$$

$$= 3^{\sqrt{2x+1}}\left(\frac{\sqrt{2x+1}}{\ln 3}\right) - \int \frac{3^{\sqrt{2x+1}}}{\ln 3}\left(\frac{1}{\sqrt{2x+1}}\right)dx$$

$$= 3^{\sqrt{2x+1}}\left(\frac{\sqrt{2x+1}}{\ln 3}\right) - \frac{1}{(\ln 3)^2}\int 3^{\sqrt{2x+1}} \bullet \frac{1n3}{\sqrt{2x+1}}\,dx$$

$$= 3^{\sqrt{2x+1}}\left(\frac{\sqrt{2x+1}}{\ln 3}\right) - \frac{1}{(\ln 3)^2}3^{\sqrt{2x+1}} + C$$

$$= \frac{3^{\sqrt{2x+1}}}{\ln 3}\left[\sqrt{2x+1} - \frac{1}{\ln 3}\right] + C \text{ (shown)}$$

(b) $\int \frac{x}{\sqrt{1-x^4}}\left[\sin^{-1}\left(x^2\right)\right]^3 dx = \frac{1}{2}\int \frac{2x}{\sqrt{1-\left(x^2\right)^2}}\left[\sin^{-1}\left(x^2\right)\right]^3 dx$

$$= \frac{1}{2}\frac{\left[\sin^{-1}\left(x^2\right)\right]^4}{4} + C = \frac{\left[\sin^{-1}\left(x^2\right)\right]^4}{8} + C \text{ (shown)}$$

4(a) $\frac{d}{dx}\left[\ln\left(\tan^3 2x\right)\right] = \frac{\left(3\tan^2 2x\right)\left(2\sec^2 2x\right)}{\tan^3 2x} = \frac{6\sec^2 2x}{\tan 2x}$

$$= 6\left(\frac{1}{\cos^2 2x}\right)\left(\frac{\cos 2x}{\sin 2x}\right) = \frac{6}{\sin 2x \cos 2x}$$

$$= \frac{12}{2\sin 2x \cos 2x} = \frac{12}{\sin 4x} \quad \text{(shown)}$$

$$\int \frac{\ln\left(\tan^3 2x\right)}{\sin 4x}dx = \frac{1}{12}\int\left(\frac{12}{\sin 4x}\right)\ln\left(\tan^3 2x\right)dx = \frac{1}{12}\frac{\left[\ln\left(\tan^3 2x\right)\right]^2}{2} + C$$

$$= \frac{\left[\ln\left(\tan^3 2x\right)\right]^2}{24} + C \quad \text{(shown)}$$

(b) For $0 \le x \le \pi$, $y = \sin 2x \sin^2\left(\cos^2 x\right)$ comprises two halves which are the reflections of each other, where $y > 0$ for $0 \le x \le \frac{\pi}{2}$, and $y < 0$ for $\frac{\pi}{2} \le x \le \pi$ due to the periodicity factor $\sin 2x$. $\therefore \int_0^\alpha \sin 2x \sin^2\left(\cos^2 x\right)dx = 0$ for $\alpha = \pi$. (shown)

(It can be understood that the positive area under the curve $y = \sin 2x \sin^2\left(\cos^2 x\right)$ for $0 \le x \le \frac{\pi}{2}$ cancels out the " negative area" under the same curve for $\frac{\pi}{2} \le x \le \pi$.)

5. $E(X) = 150p$, $Var(X) = 150p(1-p)$

Since $75Var(X) = 2[E(X)]^2$, then $75 \times 150p(1-p) = 2[150p]^2$

$11250p(1-p) = 45000p^2 \rightarrow p(1-p) = 4p^2$

$p - p^2 = 4p^2$

$5p^2 - p = 0$

$p(5p-1) = 0 \Rightarrow p = 0$ (NA) or $p = \frac{1}{5}$ (shown)

(i) $X \sim B\left(150, \frac{1}{5}\right)$

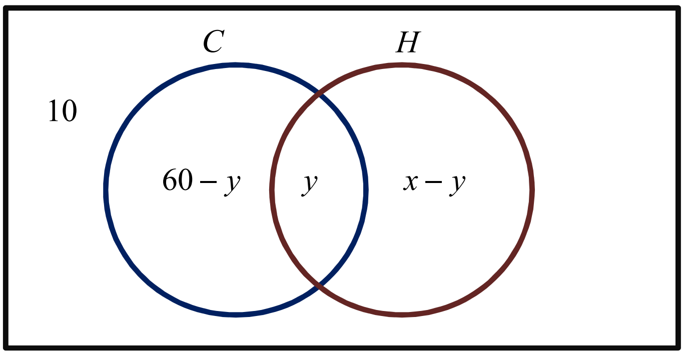
$P(X < 28) = P(\le 27) = 0.3104$ (shown)

(ii) $P(X \le n) > 0.6$

It is noted that $P(X \le 30) = 0.5487$, while $P(X \le 31) = 0.6272 > 0.6$

$\therefore$ Minimum value of $n = 31$. (shown)

6(i) Let y denote the number of buns which contained both ham and cheese.



P(random bun contained ham | bun contained cheese) = 0.75

P(random bun contained both ham and cheese) / P (bun contained cheese) = 0.75

$$\frac{\left(\frac{y}{100}\right)}{\left(\frac{60}{100}\right)} = 0.75 \Rightarrow y = 45$$

Since it is also known that $(60 - y) + y + (x - y) = 90$, substituting in $y = 45$,

$60 + x - 45 = 90 \Rightarrow x = 75$ (shown)

P (random bun contained ham) = $\frac{75}{100} = 0.75$

P (random bun contained cheese) = $\frac{60}{100} = 0.6$

Based on the earlier given conditional probability,

P(random bun contained both ham and cheese) $= 0.75 \times 0.6 = 0.45$

Since P (random bun contained ham)$\times$ P (random bun contained cheese)

also $= 0.75 \times 0.6 = 0.45$, C and H are **independent** events. (shown)

(ii) P(bun contained only cheese) = $\frac{60-45}{100} = 0.15$

P(bun contained both cheese and ham) $= \frac{45}{100} = 0.45$

Required probability $= \frac{3!}{2!}(0.15)^2(0.45) = 0.0304$ (shown)

(iii) Out of the 100 buns, 45 of them were contaminated.

Number of buns consumed which were contaminated $= 0.3 \times 60 = 18$

Number of buns which were contaminated yet unconsumed = $45 - 18 = 27$

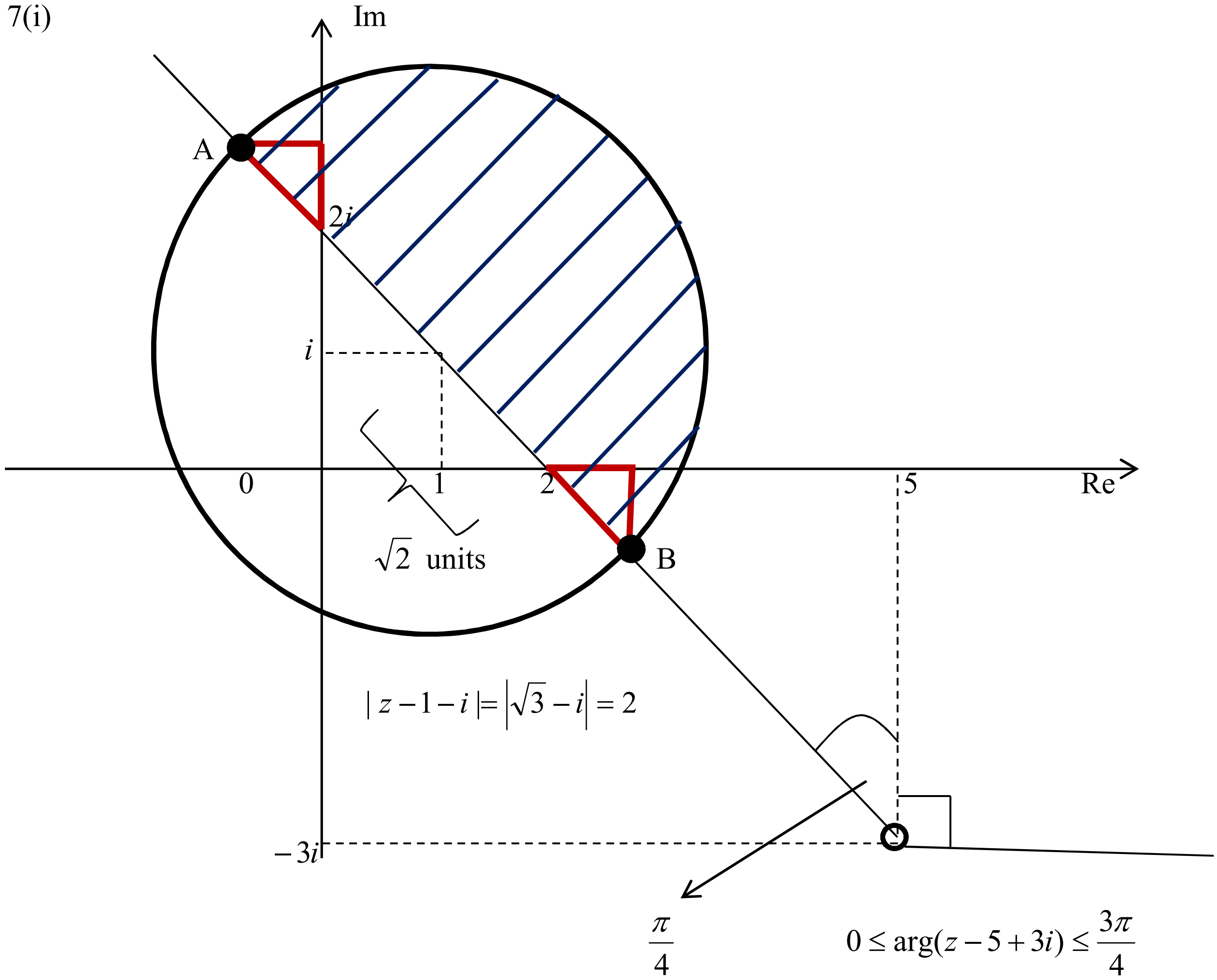
Number of unconsumed buns = $100 - 60 = 40$

P (random bun is contaminated | buns were not consumed)

= P (random bun is contaminated yet unconsumed) / P(buns were not consumed)

$= \frac{\left(\frac{27}{100}\right)}{\left(\frac{40}{100}\right)} = 0.675$ (shown)

7(i)



(ii) Let the adjacent/opposite sides of **both congruent small red right-angled triangles** be a.

Then the hypotenuse of each of these triangles would be given by $\sqrt{2}a$.

Hence, $\sqrt{2}a+\sqrt{2}=2 \Rightarrow \sqrt{2}(a+1)=2$

$$a+1=\sqrt{2}$$

$$a=\sqrt{2}-1$$

$\arg(z)_{\max}$ occurs at point A given by $-\left(\sqrt{2}-1\right)+\left(2+\sqrt{2}-1\right)(i)$

$$=\left(1-\sqrt{2}\right)+\left(\sqrt{2}+1\right)(i)$$

$\arg(z)_{\min}$ occurs at point B given by $\left(2+\sqrt{2}-1\right)-\left(\sqrt{2}-1\right)(i)$

$$=\left(\sqrt{2}+1\right)+\left(1-\sqrt{2}\right)(i) \quad \text{(shown)}$$

8(i) $(a-b)\bullet(a-b)=|a-b|^2$

Also, $(a-b)\bullet(a-b)=|a|^2-2|a||b|\cos\alpha+|b|^2$

$$=1-2\cos\alpha+1=2-2\cos\alpha \quad (\because \ |a|=|b|=1)$$

Similarly, $(a+b)\bullet(a+b)=|a+b|^2$

Also, $(a+b)\bullet(a+b)=|a|^2+2|a||b|\cos\alpha+|b|^2$

$$=1+2\cos\alpha+1=2+2\cos\alpha \quad \text{(shown)}$$

(ii) $|a+b|=3|a-b| \Rightarrow \ 2+2\cos\alpha=3(2-2\cos\alpha)$

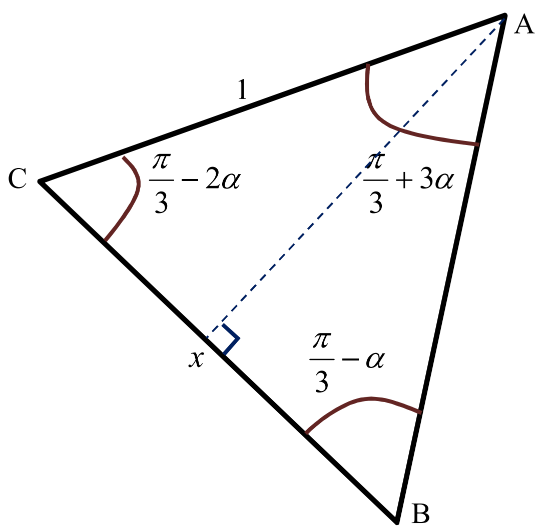
$$2+2\cos\alpha=6-6\cos\alpha$$

$$8\cos\alpha=4$$

$$\cos\alpha=\frac{1}{2}$$

$$\therefore \ \alpha=\frac{\pi}{3} \text{ (shown)}$$

9.



Shortest distance from A to BC

$$=(1)\sin\left(\frac{\pi}{3}-2\alpha\right)$$

$$= \sin\frac{\pi}{3}\cos 2\alpha - \cos\frac{\pi}{3}\sin 2\alpha$$

$$= \sin\frac{\pi}{3}\left(1 - 2\sin^2\alpha\right) - \cos\frac{\pi}{3}\left(2\sin\alpha\cos\alpha\right)$$

$$= \frac{\sqrt{3}}{2}\left(1 - 2\sin^2\alpha\right) - \frac{1}{2}\left(2\sin\alpha\cos\alpha\right)$$

$$= \frac{\sqrt{3}}{2}\left(1 - 2\sin^2\alpha\right) - \sin\alpha\cos\alpha$$

$$\approx \frac{\sqrt{3}}{2}\left(1 - 2\alpha^2\right) - (\alpha)\left(1 - \frac{\alpha^2}{2}\right)$$

$$\approx \frac{\sqrt{3}}{2} - \alpha - \sqrt{3}\alpha^2 \quad \text{(ignoring } \alpha^3 \text{ and higher powers of } \alpha\text{)} \qquad \text{(shown)}$$

By the sine rule, $\dfrac{1}{\sin\left(\dfrac{\pi}{3} - \alpha\right)} = \dfrac{x}{\sin\left(\dfrac{\pi}{3} + 3\alpha\right)}$

$$\sin\left(\frac{\pi}{3} + 3\alpha\right) = x\sin\left(\frac{\pi}{3} - \alpha\right)$$

$$\sin\frac{\pi}{3}\cos 3\alpha + \cos\frac{\pi}{3}\sin 3\alpha = x\left[\sin\frac{\pi}{3}\cos\alpha - \cos\frac{\pi}{3}\sin\alpha\right]$$

$$\sin\frac{\pi}{3}\left(4\cos^3\alpha - 3\cos\alpha\right) + \cos\frac{\pi}{3}\left(3\sin\alpha - 4\sin^3\alpha\right) = x\left[\sin\frac{\pi}{3}\cos\alpha - \cos\frac{\pi}{3}\sin\alpha\right]$$

$$\frac{\sqrt{3}}{2}\left(4\cos^3\alpha - 3\cos\alpha\right) + \frac{1}{2}\left(3\sin\alpha - 4\sin^3\alpha\right) = x\left[\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right]$$

$$\sqrt{3}\left(4\cos^3\alpha - 3\cos\alpha\right) + \left(3\sin\alpha - 4\sin^3\alpha\right) = x\left[\sqrt{3}\cos\alpha - \sin\alpha\right] \text{-------} (1)$$

When α is sufficiently small such that α^2 and higher powers of α are ignored,

$\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$.

Substituting these into (1), we get the following approximation:

$$\sqrt{3}\left[4(1)^3 - 3\right] + (3\alpha) = x\left(\sqrt{3} - \alpha\right)$$

$$x = \frac{\sqrt{3}+3\alpha}{\sqrt{3}-\alpha} = \frac{\sqrt{3}+3\alpha}{\sqrt{3}-\alpha}\left(\frac{\sqrt{3}+\alpha}{\sqrt{3}+\alpha}\right) = \frac{3+\sqrt{3}\alpha+3\sqrt{3}\alpha+3\alpha^2}{3-\alpha^2} \approx \frac{3+4\sqrt{3}\alpha}{3}$$

(once again ignoring α^2)

$\therefore \quad 3x = 3+4\sqrt{3}\alpha$ (shown)