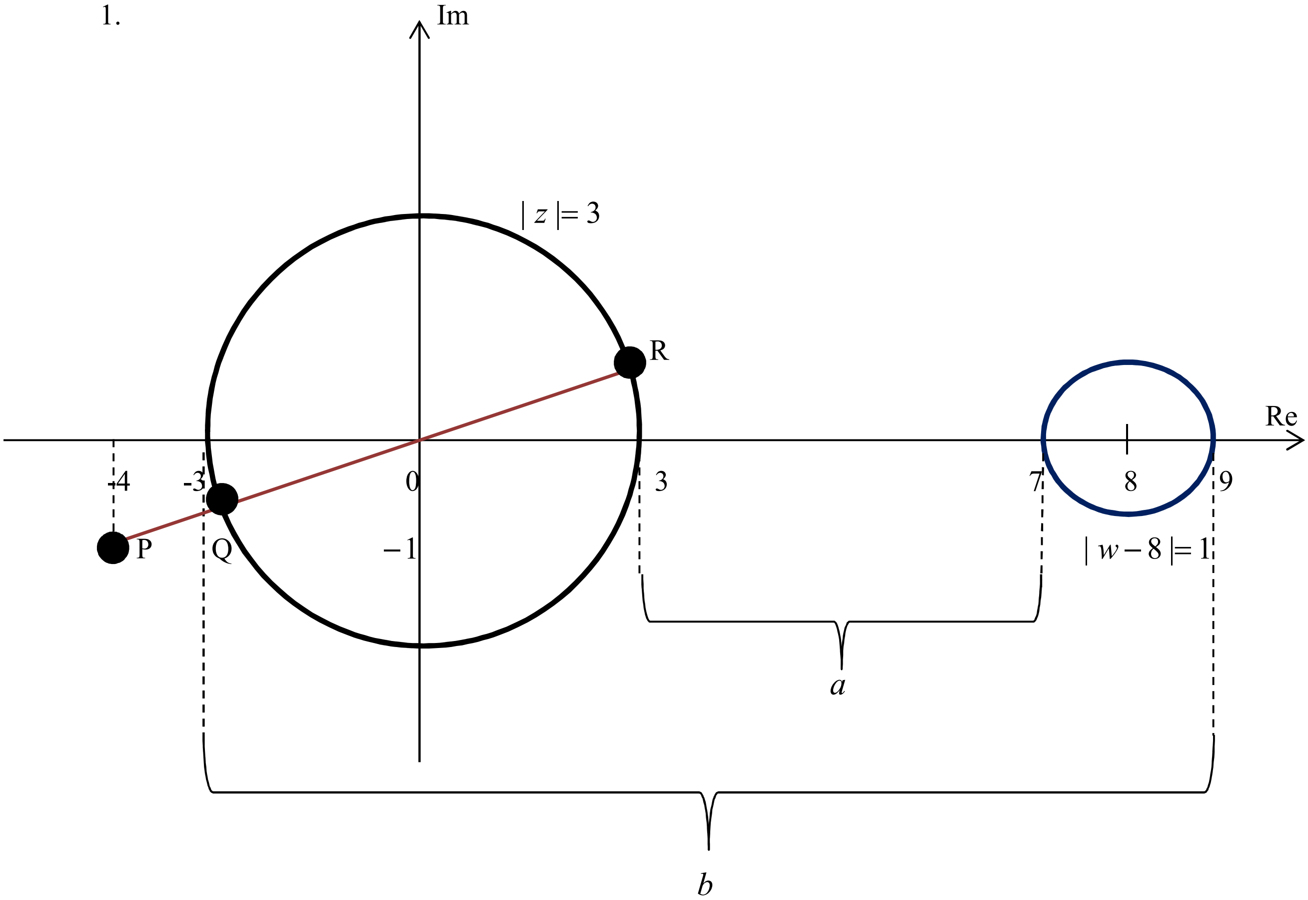
Additional Complex Number Problems 2 Solutions

1.



The **minimum** value of $|z+c|$, ie $|z+4+i|$ is given by the length of the chord $PQ = OP - OQ$ $= \underline{\sqrt{17}-3\text{ units}}$; the **maximum** value of $|z+c|$, ie $|z+4+i|$ is given by the length of the chord $PR = OP + OR = \underline{\sqrt{17}+3\text{ units}}$ (shown)

The **minimum** and **maximum** values of $|w-z|$ are $a = 7-3 = \underline{4\text{ units}}$ and $b = 9+3 = \underline{12\text{ units}}$ respectively. (shown)

2(i) $\text{Re}(z) = x = 1+\sqrt{2}\cos\theta \Rightarrow \cos\theta = \dfrac{x-1}{\sqrt{2}}$ ----------(1)

$\text{Im}(z) = y = -1+\sqrt{2}\sin\theta \Rightarrow \sin\theta = \dfrac{y+1}{\sqrt{2}}$ ----------(2)

$(1)^2+(2)^2: \left(\dfrac{x-1}{\sqrt{2}}\right)^2 + \left(\dfrac{y+1}{\sqrt{2}}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

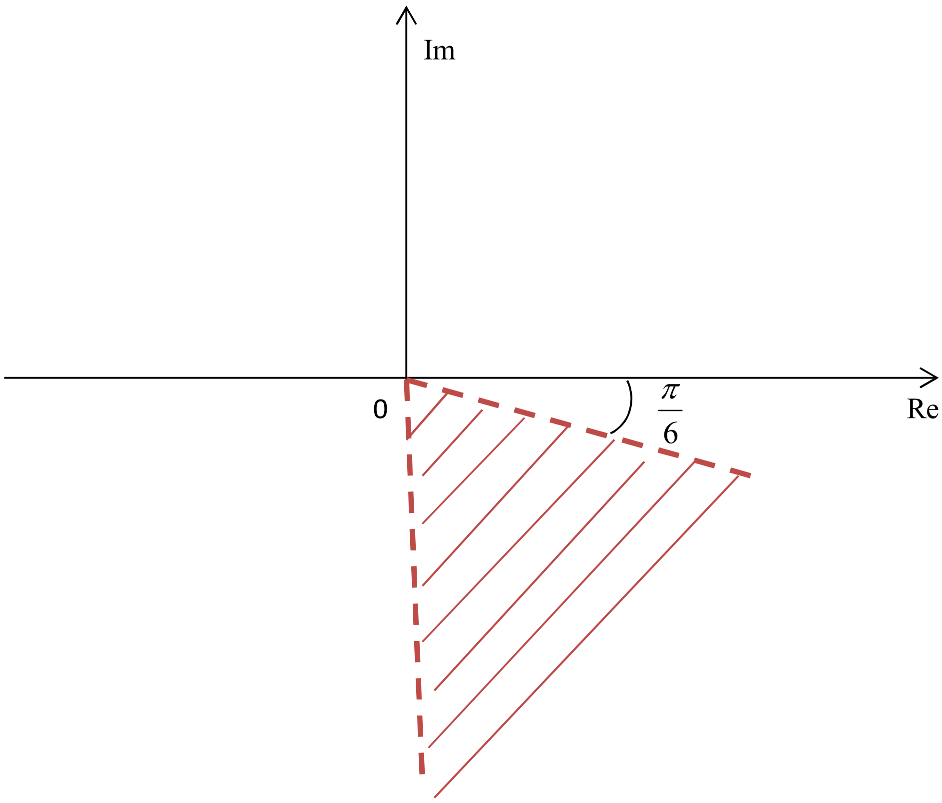
$$(x-1)^2+(y+1)^2=2$$

Hence, the variable locus z denotes a circle of radius $\sqrt{2}$ units centered at $1-i$. (shown)

(ii) $\frac{\pi}{6}<\arg\left(\frac{5}{w}\right)<\frac{\pi}{2} \to \frac{\pi}{6}<\arg(5)-\arg(w)<\frac{\pi}{2}$

$$\frac{\pi}{6}<-\arg(w)<\frac{\pi}{2}$$

$$\therefore\ -\frac{\pi}{2}<\arg(w)<-\frac{\pi}{6}$$



3(i) $z=\cos\theta+i\sin\theta=e^{i\theta}$

$$\frac{z^2-1}{z^2+1}=\frac{e^{i(2\theta)}-1}{e^{i(2\theta)}+1}=\frac{e^{i\theta}(e^{i\theta}-e^{-i\theta})}{e^{i\theta}(e^{i\theta}+e^{-i\theta})}=\frac{2i\sin\theta}{2\cos\theta}=i\tan\theta \text{ (shown)}$$

(ii) Let $z=x+iy$, then $|z|=x^2+y^2=1$

$$w=\frac{1}{1-z}=\frac{1}{1-(x+iy)}=\frac{1}{(1-x)-iy}$$

$$=\left[\frac{1}{(1-x)-iy}\right]\left[\frac{(1-x)+iy}{(1-x)+iy}\right]$$

$$=\frac{(1-x)+iy}{(1-x)^2+y^2}=\frac{(1-x)+iy}{1-2x+x^2+y^2}=\frac{(1-x)+iy}{1-2x+1}$$

$$=\frac{(1-x)+iy}{2-2x}=\frac{1-x}{2-2x}+i\frac{y}{2-2x}=\frac{1}{2}+i\frac{y}{2-2x}$$

Hence, $\operatorname{Re}(w)=\frac{1}{2}$ (shown)

4. $z^6=-1-i=\sqrt{2}e^{i\left(2k\pi+\frac{\pi}{4}\right)}$, $k=0,\ 1, 2, 3, 4, 5$

$$z=2^{\frac{1}{12}}e^{\frac{i}{6}\left(2k\pi+\frac{\pi}{4}\right)}$$

ie $z=2^{\frac{1}{12}}e^{i\frac{\pi}{24}},\ 2^{\frac{1}{12}}e^{i\frac{9\pi}{24}},\ 2^{\frac{1}{12}}e^{i\frac{17\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{23\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{15\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{3\pi}{24}}$ (shown)

$$w^6=1+i=-(-1-i)$$

$$\frac{w^6}{(-1)}=-1-i$$

$$\frac{w^6}{i^6}=-1-i$$

$$\left(\frac{w}{i}\right)^6=-1-i$$

Let $z=\frac{w}{i}$, then $w=iz$

The roots of the equation $w^6=1+i$ can be obtained by simply **multiplying every single root** of the equation $z^6=-1-i$ by $\underline{i=e^{i\frac{\pi}{2}}}$.

$\therefore\ w=2^{\frac{1}{12}}e^{i\frac{13\pi}{24}},\ 2^{\frac{1}{12}}e^{i\frac{21\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{19\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{11\pi}{24}},\ 2^{\frac{1}{12}}e^{-i\frac{\pi}{8}},\ 2^{\frac{1}{12}}e^{i\frac{3\pi}{8}}$ (shown)

5(i) Let $z = x + iy$, then $|z| \geq |z - 4i|$ becomes

$|x + iy| \geq |x + iy - 4i|$

$\sqrt{x^2 + y^2} \geq \sqrt{x^2 + (y-4)^2}$

Squaring both sides,

$x^2 + y^2 \geq x^2 + (y-4)^2$

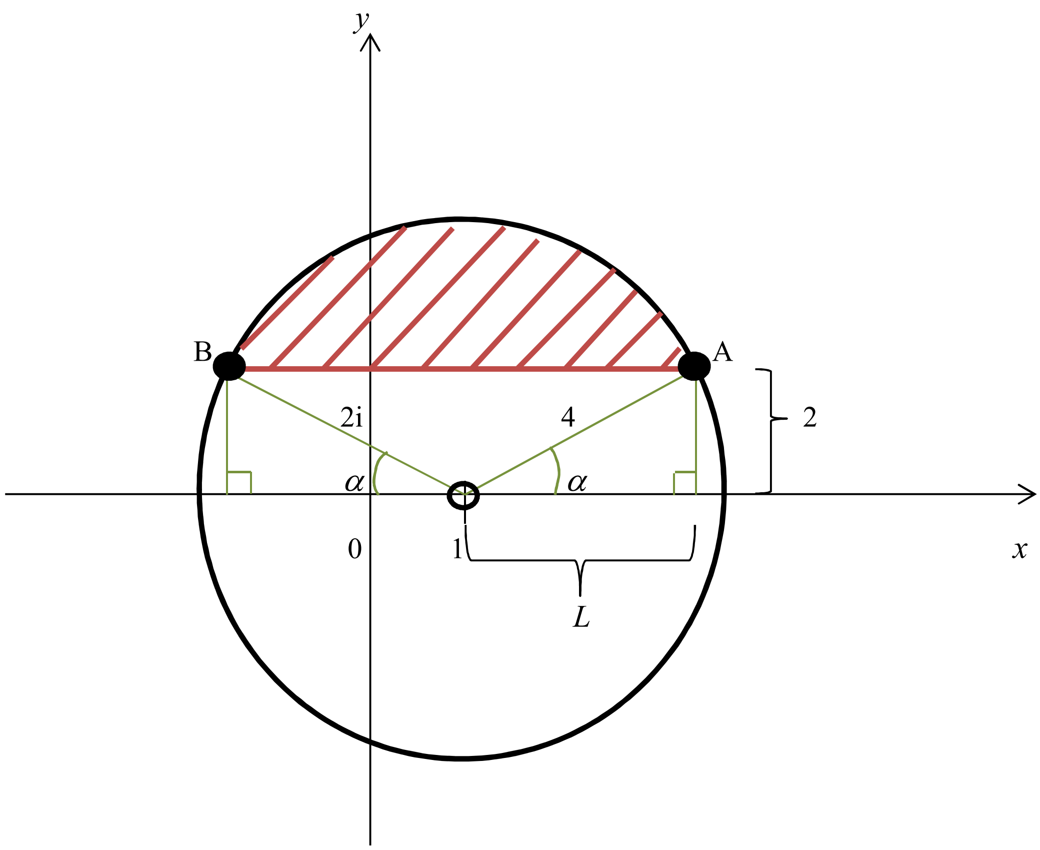
$y^2 \geq (y-4)^2$

$y^2 \geq y^2 - 8y + 16$

$8y \geq 16 \Rightarrow y \geq 2$

Since $\mathrm{Im}(z) = y$, clearly $\mathrm{Im}(z) \geq 2$ (shown)

(ii)



$\alpha = \sin^{-1}\left(\frac{2}{4}\right) = \frac{\pi}{6}$

Thus, $\frac{\pi}{6} \leq \arg(z-1) \leq \pi - \frac{\pi}{6}$, ie $\frac{\pi}{6} \leq \arg(z-1) \leq \frac{5\pi}{6}$ (shown)

(iii) $L = 4\cos\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ units

Hence, the required point (denoted by A) is given by $\left(1+2\sqrt{3}\right)+2i$. (shown)

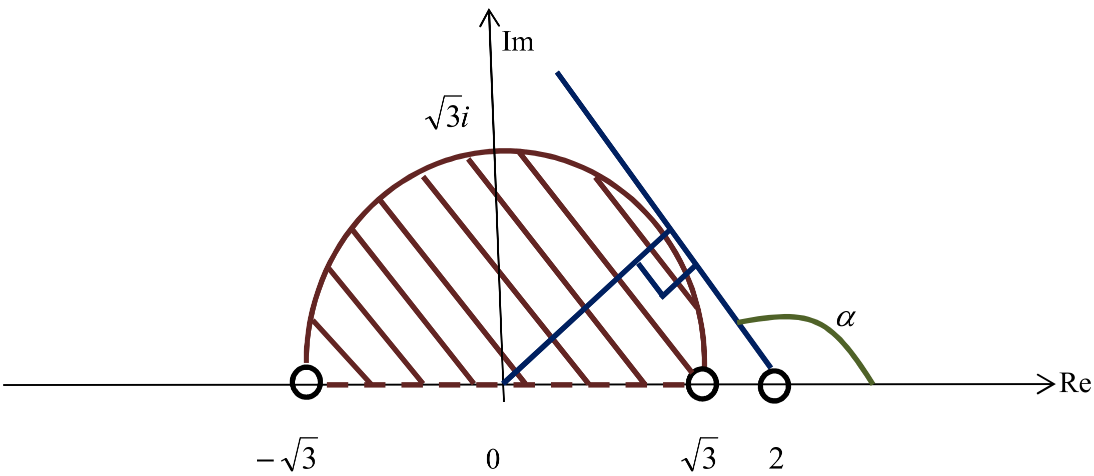
6(i) $|p| = 1, \quad \arg(p) = 2\theta$ (shown)

$p^5 = \left(e^{i2\theta}\right)^5 = e^{i10\theta} = \cos(10\theta) + i\sin(10\theta)$

Since p^5 is purely imaginary, $\cos(10\theta) = 0$, ie

$$10\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2} \quad \Rightarrow \theta = \frac{\pi}{20}, \frac{3\pi}{20}, \frac{\pi}{4}, \frac{7\pi}{20}, \frac{9\pi}{20} \text{ (shown)}$$

(ii)



$$\alpha = \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Hence, $\frac{2\pi}{3} < \arg(z-2) < \pi$ (shown)