Additional Calculus Problems 2 (Integration And Applications)

Solutions

1 (a) $u = \frac{1}{x} \to x = \frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}du$; when $x = 1,\ u = 1;\ x = \frac{1}{2},\ u = 2$

$$\text{Hence, } \int_{\frac{1}{2}}^{1} \frac{1}{x(5x^2+2x+1)}dx = \int_{2}^{1} (u)\left[\frac{1}{\frac{5}{u^2}+\frac{2}{u}+1}\right]\left(-\frac{1}{u^2}\right)du = \int_{1}^{2}(u)\left[\frac{1}{5+2u+u^2}\right]du$$

$$= \int_{1}^{2} \frac{\frac{1}{2}(2u+2)-1}{5+2u+u^2}du = \frac{1}{2}\int_{1}^{2}\frac{2u+2}{5+2u+u^2}du - \int_{1}^{2}\frac{1}{5+2u+u^2}du =$$

$$= \frac{1}{2}\left[\ln\left|5+2u+u^2\right|\right]_1^2 - \int_{1}^{2}\frac{1}{(u+1)^2+2^2}du$$

$$= \frac{1}{2}\left[\ln\left|5+2u+u^2\right|\right]_1^2 - \frac{1}{2}\left[\tan^{-1}\left(\frac{u+1}{2}\right)\right]_1^2$$

$$= \frac{1}{2}(\ln 13 - \ln 8) - \frac{1}{2}\left[\tan^{-1}\left(\frac{3}{2}\right) - \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{13}{8}\right) - \tan^{-1}\left(\frac{3}{2}\right) + \frac{\pi}{4}\right] \quad \text{(shown)}$$

(b) $\int \left(x - e^{\cos x}\right)\sin x\ dx = \int x\sin x\ dx + \int(-\sin x)e^{\cos x}dx$

$$= x(-\cos x) + \int \cos x\ dx + e^{\cos x}$$

$$= -x\cos x + \sin x + e^{\cos x} + C \quad \text{(shown)}$$

2 (a) Since $\frac{dy}{dx} = -\frac{60t}{\left(t^2-1\right)^2} < 0$ for $t > 1$, the curve C when plotted in the Cartesian plane will be **strictly** decreasing.

$\therefore$ Area $= \int_{2}^{10} x\,dy = \int_{4}^{2} x\frac{dy}{dt}dt = \int_{4}^{2}(t-1)\left[-\frac{60t}{\left(t^2-1\right)^2}\right]dt$ (when $y = 10,\ t = 2;\ y = 2,\ t = 4$)

$$= 60\int_2^4 \frac{t(t-1)}{\left(t^2-1\right)^2}\,dt = 60\int_2^4 \frac{t(t-1)}{\left[(t+1)(t-1)\right]^2}\,dt = 60\int_2^4 \frac{t}{(t+1)^2(t-1)}\,dt$$

$$= 60\int_2^4 \frac{-\frac{1}{4}}{t+1} + \frac{\frac{1}{2}}{(t+1)^2} + \frac{\frac{1}{4}}{t-1}\,dt = \int_2^4 \frac{-15}{t+1} + \frac{30}{(t+1)^2} + \frac{15}{t-1}\,dt$$

$$= \left[-15\ln|t+1| - 30\left(\frac{1}{t+1}\right) + 15\ln|t-1|\right]_2^4$$

$$= \left(-15\ln 5 - 6 + 15\ln 3\right) - \left(-15\ln 3 - 10\right) = 4 + 15\ln\left(\frac{9}{5}\right) \text{ sq units (shown)}$$

(b) When the two parabolas cut at point A, $x = (x^2-2)^2 - 2 \Rightarrow x = -1,\ y = 1$

When $x = (y-2)^2 - 2$ cuts the y-axis at point B, $x = 0,\ y = 2-\sqrt{2}$

Volume formed $= \pi\int_0^1 y\,dy - \pi\int_{2-\sqrt{2}}^1 \left[(y-2)^2 - 2\right]^2 dy$

$= 0.5\pi - 0.1503\pi = 1.099$ cubic units (shown)

3. $\dfrac{d}{dx}\ln\left(\ln x^{2x}\right) = \dfrac{d}{dx}\ln\left(x\ln x^2\right) = \dfrac{x\left(\frac{2x}{x^2}\right) + \ln\left(x^2\right)}{x\ln\left(x^2\right)} = \dfrac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}$ (shown)

$$\int \ln\left(\ln x^{2x}\right)\frac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}dx = \ln\left(\ln x^{2x}\right)\bullet\ln\left(\ln x^{2x}\right) - \int \ln\left(\ln x^{2x}\right)\frac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}dx + C$$

$$\therefore\ 2\int \ln\left(\ln x^{2x}\right)\frac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}dx = \ln\left(\ln x^{2x}\right)\bullet\ln\left(\ln x^{2x}\right) + C = \left[\ln\left(\ln x^{2x}\right)\right]^2 + C$$

$$\Rightarrow \int \ln\left(\ln x^{2x}\right)\frac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}dx = \frac{1}{2}\left[\ln\left(\ln x^{2x}\right)\right]^2 + D \text{ where } D = \frac{1}{2}C \text{ (shown)}$$

[**Alternatively**, it can be recognised that $\int \ln\left(\ln x^{2x}\right)\frac{2+\ln\left(x^2\right)}{x\ln\left(x^2\right)}dx$ is of the form $\int f(x)[f'(x)]^1 dx$,

whereby solving the integral gives $\dfrac{[f(x)]^2}{2} + D = \dfrac{1}{2}\left[\ln\left(\ln x^{2x}\right)\right]^2 + D$]

4. $x=\tan\theta \to dx=\sec^2\theta$

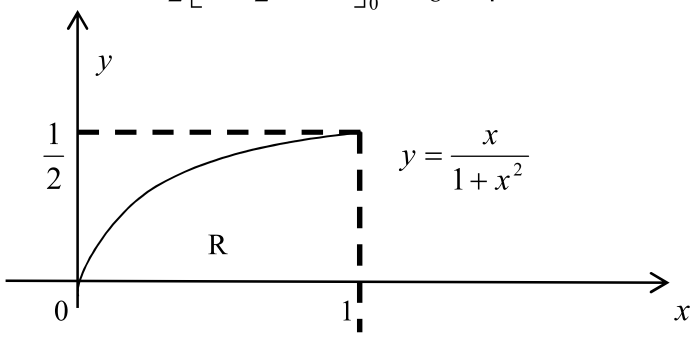
When $x=1, \theta=\frac{\pi}{4}; \quad x=0,\ \theta=0$

$$\int_0^1 \frac{x^2}{(1+x^2)^2}dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \tan^2\theta\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1-\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}-\frac{1}{4} \text{ (shown)}$$



$$\text{Volume generated } = \pi\left(\frac{1}{2}\right)^2(1) - \pi\int_0^1\left(\frac{1}{2}-y\right)^2 dx$$

$$= \frac{\pi}{4} - \pi\int_0^1 \frac{1}{4} - y + y^2 dx$$

$$= \frac{\pi}{4} - \frac{1}{4}\pi\int_0^1 dx + \pi\int_0^1 \frac{x}{1+x^2}dx - \pi\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}dx$$

$$= \frac{\pi}{4} - \frac{\pi}{4} + \frac{\pi}{2}\left[\ln\left|1+x^2\right|\right]_0^1 - \pi\left(\frac{\pi}{8}-\frac{1}{4}\right)$$

$$= \frac{\pi}{2}\ln 2 - \frac{\pi^2}{8} + \frac{\pi}{4} \text{ cubic units where } a=4 \text{ and } b=2 \text{ (shown)}$$

5(a) $\int \frac{5-2x}{\sqrt{2-4x-x^2}}dx = \int \frac{(-4-2x)+9}{\sqrt{2-4x-x^2}}dx = \int \frac{-4-2x}{\sqrt{2-4x-x^2}}dx + \int \frac{9}{\sqrt{2-4x-x^2}}dx$

$$= \int (-4-2x)(2-4x-x^2)^{-\frac{1}{2}}dx + 9\int \frac{1}{\sqrt{6-(x+2)^2}}dx$$

$$= 2\sqrt{2-4x-x^2} + 9\sin^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + C \quad \text{(shown)}$$

(b) $\int_0^N x^2e^{-x}dx = \left[(x^2)(-e^{-x})\right]_0^N + 2\int_0^N e^{-x}(x)dx = N^2e^{-N} + 2\left[(-e^{-x})(x)\right]_0^N + 2\int_0^N e^{-x}dx$

$$= N^2e^{-N} - 2Ne^{-N} + 2\left[-e^{-x}\right]_0^N = N^2e^{-N} - 2Ne^{-N} - 2e^{-N} + 2 \quad \text{(shown)}$$

When $n \to \infty,\ e^{-N} \to 0$

Hence, $\lim_{N\to\infty}\int_0^N x^2e^{-x}dx = 2$ (shown)

(c) $u = x+4 \Rightarrow du = dx$ When $\underline{x=5,\ u=9};\ \underline{x=0,\ u=4};\ \underline{x=-4,\ u=0}$

$$\int_{-4}^{5} |x|\sqrt{x+4}\ dx = \int_0^5 x\sqrt{x+4}\ dx - \int_{-4}^{0} x\sqrt{x+4}\ dx$$

$$= \int_4^9 (u-4)(\sqrt{u})du - \int_0^4 (u-4)(\sqrt{u})du$$

$$= \int_4^9 u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\ du - \int_0^4 u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\ du$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_4^9 - \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_0^4 = \frac{634}{15} \quad \text{(shown)}$$

6. $\int_0^{2\pi} e^x \cos mx\ dx = \left[e^x\cos mx\right]_0^{2\pi} + m\int_0^{2\pi} e^x \sin mx\ dx$

$$= e^{2\pi} - 1 + m\left[e^x \sin mx\right]_0^{2\pi} - m^2\int_0^{2\pi} e^x\cos mx\ dx$$

$$= e^{2\pi} - 1 - m^2 \int_0^{2\pi} e^x \cos mx \, dx$$

$$\therefore (1+m^2)\int_0^{2\pi} e^x \cos mx \, dx = e^{2\pi} - 1 \Rightarrow \int_0^{2\pi} e^x \cos mx \;\; dx \, \frac{1}{m^2+1}\left(e^{2\pi}-1\right) \quad \text{(shown)}$$

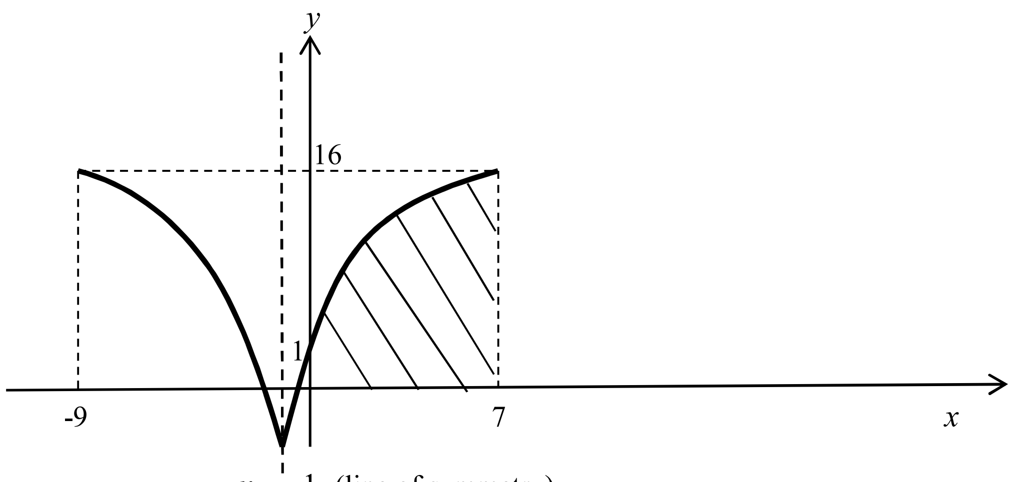
$$\cos(A+B) + \cos(A-B) = 2\cos\left[\frac{A+B+A-B}{2}\right]\cos\left[\frac{A+B-A+B}{2}\right] = 2\cos A \cos B$$

$$\int_0^{2\pi} e^x \cos x \cos 6x \; dx = \frac{1}{2}\int_0^{2\pi} e^x\left(\cos 7x + \cos 5x\right)dx = \frac{1}{2}\int_0^{2\pi} e^x \cos 7x \; dx + \frac{1}{2}\int_0^{2\pi} e^x \cos 5x \; dx$$

$$= \frac{1}{2}\left(\frac{1}{7^2+1}\right)\left(e^{2\pi}-1\right) + \frac{1}{2}\left(\frac{1}{5^2+1}\right)\left(e^{2\pi}-1\right)$$

$$= \frac{1}{100}\left(e^{2\pi}-1\right) + \frac{1}{52}\left(e^{2\pi}-1\right) = \frac{19}{650}\left(e^{2\pi}-1\right) \;\; \text{(shown)}$$

7.



$x = -1$ (line of symmetry)

$$\text{Area} = \int_0^7 y dx = \int_1^2 y \frac{dx}{dt} dt = \int_1^2 \left(5t^2 - 4\right)\left(3t^2\right)dt = \int_1^2 15t^4 - 12t^2 dt$$

$$= \left[3t^5 - 4t^3\right]_1^2 = \text{65 sq units (shown)}$$