Extreme Problem 7 Solutions

(i) Let the distance from the top surface of the cylindrical tank to the vertex of the cone structure be x. Then $\tan 60^\circ = \frac{r}{x} \Rightarrow x = r\cot 60^\circ = \frac{r}{\sqrt{3}}$

Height of cylindrical tank is therefore $h - x = h - \frac{r}{\sqrt{3}}$ and

Volume of tank $V_T = \pi\, r^2 \left(h - \frac{r}{\sqrt{3}} \right) = \pi\, r^2 h - \frac{1}{\sqrt{3}} \pi\, r^3$

When the cylindrical tank has a maximum volume, $\frac{dV_T}{dr} = 0$ (Note: h is a constant)

$$2\pi\, rh - \sqrt{3}\,\pi\, r^2 = 0$$

$$\pi\, r\left(2h - \sqrt{3}r\right) = 0$$

$$\therefore \quad r = \frac{2}{\sqrt{3}}h = \frac{2\sqrt{3}}{3}h \quad \text{(shown) or } r = 0 \text{ (NA)}$$

(ii) Let the slant height of the liquid chemical occupying the inverted cone at any time instant t be l. Since the semi-vertical angle of the said inverted cone is 45°, we have $l = \sqrt{2}r$.

Also, vertical height of liquid chemical in cone shall be equivalent to r.

As such, lateral surface area of cone occupied by liquid chemical at time t is given by

$A = \pi\, r\, l = \sqrt{2}\pi\, r^2$ -----------(1)

Volume of cone $V = \frac{1}{3}\pi\, r^2 \times$ height of liquid chemical $= = \frac{1}{3}\pi\, r^3$ -----------(2)

From (1), $r = \left(\frac{A}{\sqrt{2}\pi} \right)^{\frac{1}{2}}$; substituting this into (2) gives $V = \frac{1}{3}\pi\left(\frac{A}{\sqrt{2}\pi} \right)^{\frac{3}{2}} = \frac{2^{-\frac{3}{4}}}{3\sqrt{\pi}} A^{\frac{3}{2}}$

Therefore, $\frac{dV}{dA} = \frac{3}{2} \bullet \frac{2^{-\frac{3}{4}}}{3\sqrt{\pi}} A^{\frac{1}{2}} = \frac{2^{-\frac{7}{4}}}{\sqrt{\pi}} A^{\frac{1}{2}}$

After 30 minutes, volume of liquid chemical leaked into inverted cone $= 30 \times 0.3 = 9\ m^3$

$$\frac{1}{3}\pi\ r^3 = 9 \Rightarrow r = \left(\frac{27}{\pi}\right)^{\frac{1}{3}} \quad \text{and} \quad A = \sqrt{2}\pi\left(\frac{27}{\pi}\right)^{\frac{2}{3}} = 9\sqrt{2}\pi^{\frac{1}{3}}\ m^2$$

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt} \Rightarrow\ 0.3 = \frac{2^{-\frac{7}{4}}}{\sqrt{\pi}} A^{\frac{1}{2}} \times \frac{dA}{dt}$$

Since $A = 9\sqrt{2}\pi^{\frac{1}{3}}\ m^2$, $0.3 = \frac{2^{-\frac{7}{4}}}{\sqrt{\pi}}\left(9\sqrt{2}\pi^{\frac{1}{3}}\right)^{\frac{1}{2}} \times \frac{dA}{dt}$

$$= \frac{2^{-\frac{7}{4}}}{\sqrt{\pi}}(3)(2)^{\frac{1}{4}}\pi^{\frac{1}{6}} \times \frac{dA}{dt}$$

$$= 3(2)^{-\frac{3}{2}}\pi^{-\frac{1}{3}} \times \frac{dA}{dt}$$

Hence, $\frac{dA}{dt} = \frac{3}{10} \div \frac{3}{2\sqrt{2}}\pi^{-\frac{1}{3}} = \frac{\sqrt{2}}{5}\pi^{\frac{1}{3}}\ m^2$ / min (shown)