Extreme Problem 6 Solutions

$$w = \frac{1}{z-2} = \frac{1}{(x-2)+iy} = \frac{(x-2)-iy}{(x-2)^2+y^2}$$

$$= \frac{x-2}{(x-2)^2+y^2} + i\frac{(-y)}{(x-2)^2+y^2}$$

Since $w = u + iv$, then $\frac{x-2}{(x-2)^2+y^2} = u$

$$\frac{x-2}{(x-2)^2+y^2} - 1 = u - 1$$

$$\left[\frac{x-2}{(x-2)^2+y^2} - 1\right]^2 = (u-1)^2 \text{ -------- } (1)$$

$$\frac{(-y)}{(x-2)^2+y^2} = v$$

$$\frac{(-y)}{(x-2)^2+y^2} + \frac{1}{2} = v + \frac{1}{2}$$

$$\left[\frac{(-y)}{(x-2)^2+y^2} + \frac{1}{2}\right]^2 = \left(v+\frac{1}{2}\right)^2 \text{ ---------(2)}$$

$(1)+(2):\ (u-1)^2 + \left(v+\frac{1}{2}\right)^2$

$$= \left[\frac{x-2}{(x-2)^2+y^2} - 1\right]^2 + \left[\frac{(-y)}{(x-2)^2+y^2} + \frac{1}{2}\right]^2$$

$$= \left[\frac{x-2}{(x-2)^2+y^2}\right]^2 - \frac{2(x-2)}{(x-2)^2+y^2} + 1 + \left[\frac{y}{(x-2)^2+y^2}\right]^2 - \frac{y}{(x-2)^2+y^2} + \frac{1}{4}$$

$$= \frac{(x-2)^2}{\left[(x-2)^2+y^2\right]^2} + \frac{y^2}{\left[(x-2)^2+y^2\right]^2} - \frac{2(x-2)}{(x-2)^2+y^2} - \frac{y}{(x-2)^2+y^2} + \frac{5}{4}$$

$$= \frac{(x-2)^2}{\left[(x-2)^2+y^2\right]^2} + \frac{y^2}{\left[(x-2)^2+y^2\right]^2} + \frac{-2x+4-y}{(x-2)^2+y^2} + \frac{5}{4}$$

$$= \frac{(x-2)^2 + y^2}{\left[(x-2)^2 + y^2\right]^2} + \frac{-(2x+y)+4}{(x-2)^2 + y^2} + \frac{5}{4}$$

$$= \frac{1}{(x-2)^2 + y^2} + \frac{-(2x+y)+4}{(x-2)^2 + y^2} + \frac{5}{4}$$

$$= \frac{1}{(x-2)^2 + y^2} + \frac{-5+4}{(x-2)^2 + y^2} + \frac{5}{4}$$ (substituting $2x + y = 5$)

$$= \frac{1}{(x-2)^2 + y^2} - \frac{1}{(x-2)^2 + y^2} + \frac{5}{4} = \frac{5}{4}$$

Hence, the straight line with equation $2x + y = 5$ in the $z-$ plane is transformed into a circle in the $w-$plane centred at $\left(1, -\frac{1}{2}\right)$ and radius of $\frac{\sqrt{5}}{2}$ units. (shown)