Extreme Problem 2 Solutions

A vector normal to plane $BCD = \overrightarrow{BC} \times \overrightarrow{BD} = \left[\begin{pmatrix}2\\3\\4\end{pmatrix} - \begin{pmatrix}5\\6\\4\end{pmatrix}\right] \times \left[\begin{pmatrix}6\\2\\3\end{pmatrix} - \begin{pmatrix}5\\6\\4\end{pmatrix}\right]$

$$= \begin{pmatrix}-3\\-3\\0\end{pmatrix} \times \begin{pmatrix}1\\-4\\-1\end{pmatrix} = \begin{pmatrix}3\\-3\\15\end{pmatrix} // \begin{pmatrix}1\\-1\\5\end{pmatrix} \text{ (shown)}$$

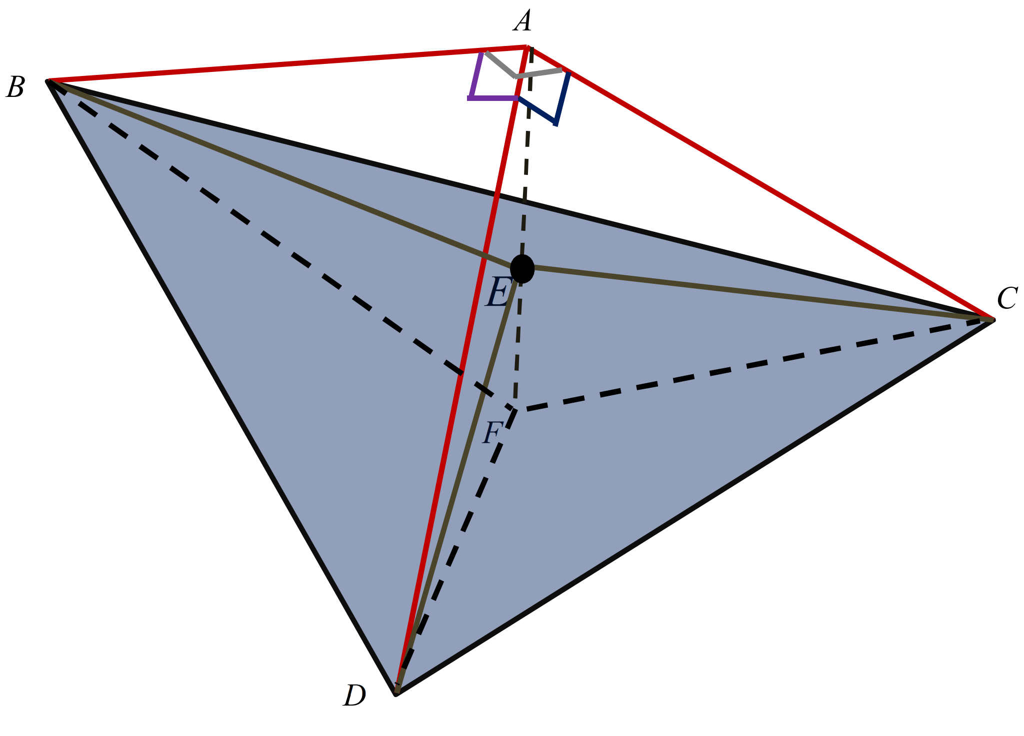
Equation of plane BCD is $x - y + z = (1)(2) + (-1)(3) + (5)(4) = 19$ (shown)

Distance of A from the plane $= \left|\overrightarrow{BA} \bullet \hat{n}\right| = \frac{1}{\sqrt{27}}\left|\begin{pmatrix}-1\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}1\\-1\\5\end{pmatrix}\right| = \frac{9}{\sqrt{27}} = \frac{9}{3\sqrt{3}} = \sqrt{3}$ units (shown)

$\overrightarrow{AB} = \begin{pmatrix}5\\6\\4\end{pmatrix} - \begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $\quad \overrightarrow{AC} = \begin{pmatrix}2\\3\\4\end{pmatrix} - \begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$, $\quad \overrightarrow{AD} = \begin{pmatrix}6\\2\\3\end{pmatrix} - \begin{pmatrix}4\\4\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

Clearly, $|\overrightarrow{AB}| = |\overrightarrow{AC}| = |\overrightarrow{AD}| = 3$ units and $\overrightarrow{AB} \bullet \overrightarrow{AC} = \overrightarrow{AC} \bullet \overrightarrow{AD} = 0$

Hence, AB, AC and AD are of equal length and are perpendicular to one another. (shown)



From the above figure, point E is located along the normal to the plane BCD **but resides above** the foot of perpendicular from A to the plane BCD (designated point F)

$$\overrightarrow{OE}-\overrightarrow{OA}=k\begin{pmatrix}1\\-1\\5\end{pmatrix}\Rightarrow\overrightarrow{OE}=\begin{pmatrix}4\\4\\2\end{pmatrix}+k\begin{pmatrix}1\\-1\\5\end{pmatrix}=\begin{pmatrix}4+k\\4-k\\2+5k\end{pmatrix}$$

$$\left|\overrightarrow{EB}\right|=3\Rightarrow\left|\begin{pmatrix}4+k\\4-k\\2+5k\end{pmatrix}-\begin{pmatrix}5\\6\\4\end{pmatrix}\right|=3 \quad (*)$$

$$\left|\begin{pmatrix}k-1\\-2-k\\5k-2\end{pmatrix}\right|=3$$

$$\left|\begin{pmatrix}k-1\\-2-k\\5k-2\end{pmatrix}\right|^2=9$$

$$(k-1)^2+(-2-k)^2+(5k-2)^2=9$$

$$k^2-2k+1+k^2+4k+4+25k^2-20k+4=9$$

$$27k^2-18k=0\rightarrow 3k(3k-2)=0$$

$k=0$ (rejected) or $k=\dfrac{2}{3}$

$$\overrightarrow{OE}=\begin{pmatrix}4\\4\\2\end{pmatrix}+\frac{2}{3}\begin{pmatrix}1\\-1\\5\end{pmatrix}=\frac{1}{3}\begin{pmatrix}14\\10\\16\end{pmatrix} \quad \text{(shown)}$$

(*) Note: we can also solve for k by setting $\left|\overrightarrow{EC}\right|=3$ or $\left|\overrightarrow{ED}\right|=3$