Let

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$$

and

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx$$

where n is an integer, $n \geq 0$. (Note that $A_n > 0$, $B_n > 0$.)

(a) Show that $nA_n = \frac{2n-1}{2} A_{n-1}$ for $n \geq 1$.

(b) Using integration by parts on A_n, or otherwise, show that

$$A_n = 2n \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx$$

for $n \geq 1$.

(c) Use integration by parts on the integral in part (b) to show that

$$\frac{A_n}{n^2} = \frac{2n-1}{n} B_{n-1} - 2B_n$$

for $n \geq 1$.

(d) Use parts (a) and (c) to show that

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right)$$

for $n \geq 1$.

(e) Show that

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$$

(f) Use the fact that $\sin x \geq \frac{2}{\pi} x$ for $0 \leq x \leq \frac{\pi}{2}$ to show that

$$B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx$$

(g) Show that

$$\int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx = \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx$$

(h) From parts (f) and (g) it follows that

$$B_n \leq \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx$$

Use the substitution $x = \frac{\pi}{2} \sin t$ in this inequality to show that

$$B_n \leq \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+3} t \, dt \leq \frac{\pi^3}{16(n+1)} A_n$$

(CONTINUES OVERLEAF)

(i) Use part (e) to deduce that

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6}$$

(j) What is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$$