

EXTREME PROBLEM 11 SOLUTIONS

$$\begin{aligned} \text{(a) } A_n &= \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \int_0^{\frac{\pi}{2}} (\cos^{2n-1} x)(\cos x) dx \\ &= \left[(\sin x)(\cos^{2n-1} x) \right]_0^{\frac{\pi}{2}} - (2n-1) \int_0^{\frac{\pi}{2}} (\cos^{2n-2} x)(-\sin x)(\sin x) dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} (\cos^{2n-2} x)(\sin^2 x) dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} (\cos^{2n-2} x)(\sin^2 x) dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} (\cos^{2n-2} x)(1 - \cos^2 x) dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} (\cos^{2n-2} x) dx - (2n-1) \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \\ &= (2n-1)A_{n-1} - (2n-1)A_n \end{aligned}$$

$$\therefore (1+2n-1)A_n = (2n-1)A_{n-1} \Rightarrow nA_n = \left(\frac{2n-1}{2} \right) A_{n-1} \quad (\text{shown})$$

$$\begin{aligned} \text{(b) } A_n &= \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \left[x(\cos^{2n} x) \right]_0^{\frac{\pi}{2}} - 2n \int_0^{\frac{\pi}{2}} x \cos^{2n-1} x (-\sin x) dx \\ &= 2n \int_0^{\frac{\pi}{2}} x(\sin x) \cos^{2n-1} x \, dx \quad (\text{shown}) \end{aligned}$$

$$(c) A_n = 2n \int_0^{\frac{\pi}{2}} x(\sin x) \cos^{2n-1} x \, dx$$

$$= 2n \left\{ \left[\left(\frac{x^2}{2} \right) (\sin x) (\cos^{2n-1} x) \right]_0^{\pi/2} - \int_0^{\pi/2} \left(\frac{x^2}{2} \right) (\cos x \cdot \cos^{2n-1} x + \sin x \cdot (2n-1) \cos^{2n-2} x \cdot (-\sin x)) dx \right\}$$

$$= -n \int_0^{\frac{\pi}{2}} (x^2) (\cos^{2n} x) - (x^2) (2n-1) \sin^2 x (\cos^{2n-2} x) \, dx$$

$$= -n \int_0^{\frac{\pi}{2}} (x^2) (\cos^{2n} x) \, dx + n(2n-1) \int_0^{\frac{\pi}{2}} x^2 (\sin^2 x) (\cos^{2n-2} x) \, dx$$

$$= -nB_n + n(2n-1) \int_0^{\frac{\pi}{2}} x^2 (\sin^2 x) (\cos^{2n-2} x) \, dx$$

$$= -nB_n + n(2n-1) \int_0^{\frac{\pi}{2}} x^2 (1 - \cos^2 x) (\cos^{2n-2} x) \, dx$$

$$= -nB_n + n(2n-1) \int_0^{\frac{\pi}{2}} x^2 (\cos^{2n-2} x) \, dx - n(2n-1) \int_0^{\frac{\pi}{2}} x^2 (\cos^{2n} x) \, dx$$

$$= -nB_n + n(2n-1)B_{n-1} - n(2n-1)B_n$$

$$= n(2n-1)B_{n-1} - 2n^2 B_n$$

Dividing $A_n = n(2n-1)B_{n-1} - 2n^2 B_n$ by n^2 on both sides thus gives

$$\frac{A_n}{n^2} = \left(\frac{2n-1}{n} \right) B_{n-1} - 2B_n \text{ (shown)}$$

$$(d) \frac{A_n}{n^2} = \left(\frac{2n-1}{n} \right) B_{n-1} - 2B_n$$

$$\frac{1}{n^2} = \left(\frac{2n-1}{n} \right) \frac{B_{n-1}}{A_n} - \frac{2B_n}{A_n}$$

$$\frac{1}{n^2} = \left(\frac{2n-1}{n} \right) \frac{B_{n-1}}{A_n} - \frac{2B_n}{A_n} \text{----- (1)}$$

Since $nA_n = \left(\frac{2n-1}{2} \right) A_{n-1}$, then (1) becomes

$$\begin{aligned} \frac{1}{n^2} &= \frac{(2n-1)B_{n-1}}{\left(\frac{2n-1}{2} \right) A_{n-1}} - \frac{2B_n}{A_n} \\ &= \frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n} = 2 \left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n} \right) \text{ (shown)} \end{aligned}$$

(e) By dummy variable exchange implication,

$$\frac{1}{k^2} = 2 \left(\frac{B_{k-1}}{A_{k-1}} - \frac{B_k}{A_k} \right)$$

$$\sum_{k=1}^n \frac{1}{k^2} = 2 \sum_{k=1}^n \left(\frac{B_{k-1}}{A_{k-1}} - \frac{B_k}{A_k} \right)$$

$$= 2 \left[\begin{array}{l} \frac{B_0}{A_0} - \frac{B_1}{A_1} \\ \frac{B_1}{A_1} - \frac{B_2}{A_2} \\ \frac{B_2}{A_2} - \frac{B_3}{A_3} \\ \vdots \\ \frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n} \end{array} \right]$$

$$\begin{aligned}
&= 2 \frac{B_0}{A_0} - 2 \frac{B_n}{A_n} = \frac{2 \int_0^{\frac{\pi}{2}} x^2 dx}{\int_0^{\frac{\pi}{2}} dx} - 2 \frac{B_n}{A_n} = \frac{2 \left[\frac{x^3}{3} \right]_0^{\frac{\pi}{2}}}{\left[x \right]_0^{\frac{\pi}{2}}} - 2 \frac{B_n}{A_n} = \frac{2 \left(\frac{\pi^3}{24} \right)}{\frac{\pi}{2}} - 2 \frac{B_n}{A_n} \\
&= \frac{\pi^2}{6} - 2 \frac{B_n}{A_n} \quad (\text{shown})
\end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad B_n &= \int_0^{\frac{\pi}{2}} x^2 (\cos^{2n} x) dx = \int_0^{\frac{\pi}{2}} x^2 (\cos^2 x)^n dx \\
&= \int_0^{\frac{\pi}{2}} x^2 (1 - \sin^2 x)^n dx \text{-----(2)}
\end{aligned}$$

Since $\sin x \geq \frac{2x}{\pi}$, then $\sin^2 x \geq \left(\frac{2x}{\pi} \right)^2 = \frac{4x^2}{\pi^2}$

Reconciling this with (2) therefore yields the inequality $B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2} \right)^n dx$ (shown)

$$\text{(g)} \quad \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2} \right)^n dx = \int_0^{\frac{\pi}{2}} x \bullet x \left(1 - \frac{4x^2}{\pi^2} \right)^n dx \quad \left[\because \int x \left(1 - \frac{4x^2}{\pi^2} \right)^n dx = -\frac{\pi^2}{8} \frac{\left(1 - \frac{4x^2}{\pi^2} \right)^{n+1}}{n+1} \right]$$

$$= -\frac{\pi^2}{8} \left[\frac{\left(1 - \frac{4x^2}{\pi^2} \right)^{n+1}}{n+1} \bullet x \right]_0^{\frac{\pi}{2}} + \frac{\pi^2}{8} \int_0^{\frac{\pi}{2}} \frac{\left(1 - \frac{4x^2}{\pi^2} \right)^{n+1}}{n+1} dx$$

$$= \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2} \right)^{n+1} dx \quad (\text{shown})$$

(h) Let $x = \frac{\pi}{2} \sin t$, then $\frac{dx}{dt} = \frac{\pi}{2} \cos t$

$$\begin{aligned} \text{Then } \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx &= \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} (1 - \sin^2 t)^{n+1} \left(\frac{\pi}{2} \cos t\right) dt \\ &= \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} (\cos^2 t)^{n+1} (\cos t) dt = \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+3} t dt \text{ -----(3)} \end{aligned}$$

For $0 \leq t \leq \frac{\pi}{2}$, $\cos^n t \geq 0$ where $n \in \mathbb{Z}^+$,

then for $0 \leq a \leq b$, where both $a, b \in \mathbb{Z}^+$, $\cos^b t \leq \cos^a t$

and $\int_0^{\frac{\pi}{2}} \cos^b t dt \leq \int_0^{\frac{\pi}{2}} \cos^a t dt$ (consider area under both curves for comparison purposes)

Naturally, $\int_0^{\frac{\pi}{2}} \cos^{2n+3} t dt \leq \int_0^{\frac{\pi}{2}} \cos^n t dt$, ie $\int_0^{\frac{\pi}{2}} \cos^{2n+3} t dt \leq A_n$ -----(4)

Since $B_n \leq \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx$,

By (3) and (4), $B_n \leq \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+3} t dt \leq \frac{\pi^3}{16(n+1)} A_n$ (shown)

(i) The inequality derived in (e) is $B_n \leq \frac{\pi^3}{16(n+1)} A_n \Rightarrow \frac{B_n}{A_n} \leq \frac{\pi^3}{16(n+1)}$

$$\sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - 2 \frac{B_n}{A_n} \geq \frac{\pi^2}{6} - 2 \left[\frac{\pi^3}{16(n+1)} \right] = \frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)}$$

Also, it is noted that an obvious consequence of the inequality derived in (e) is

$$\sum_{k=1}^n \frac{1}{k^2} < \frac{\pi^2}{6}$$

Reconciling all these therefore gives $\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^n \frac{1}{k^2} < \frac{\pi^2}{6}$ (shown)

$$(j) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} \text{ (shown)}$$