Extreme Problem 1 Solutions

$|v|$ (length of tangent)
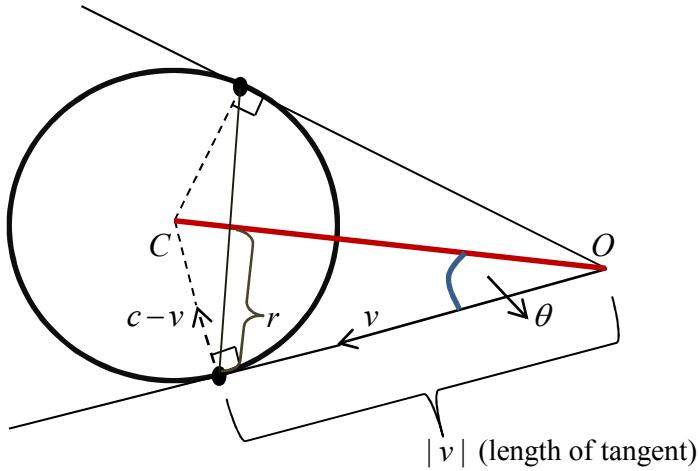
Radius of sphere $= \dfrac{|b-a|}{2}$

Let position vector of centre of sphere be c, then $c = \dfrac{a+b}{2}$

Denoting the loci of a random point on the sphere as v,

Equation of the sphere is therefore given by $(v-c)\bullet(v-c) = \left[\dfrac{|b-a|}{2}\right]^2$

$$(v-c)\bullet(v-c) = \frac{1}{4}|b-a|^2 = \frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v - 2v\bullet c + c\bullet c = \frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v - 2v\bullet c + \left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right) = \frac{1}{4}(b-a)\bullet(b-a) ---------(1)$$

If tangents can be drawn from O to the sphere, then

$$v\bullet(c-v) = 0 \Rightarrow v\bullet c = v\bullet v$$

Substituting this into (1),

$$v\bullet v - 2v\bullet v + \left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right) = \frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v = \left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right) - \frac{1}{4}(b-a)\bullet(b-a)$$

$$|v|^2 = \frac{1}{4}\left[(a+b)\bullet(a+b) - (b-a)\bullet(b-a)\right]$$

$$= \frac{1}{4}\left[(a\bullet a + 2a\bullet b + b\bullet b) - (b\bullet b - 2a\bullet b + a\bullet a)\right]$$

$$= \frac{1}{4}(4a\bullet b) = a\bullet b$$

$$\therefore \quad |v| = (a\bullet b)^{\frac{1}{2}} \quad \text{(shown)}$$

$$v\bullet c = |v||c|\cos\theta$$

Since $v\bullet c = v\bullet v$, $\cos\theta = \dfrac{v\bullet c}{|v||c|} = \dfrac{v\bullet v}{|v||c|} = \dfrac{|v|^2}{|v||c|}$

$$= \frac{|v|}{|c|} = \frac{(a\bullet b)^{\frac{1}{2}}}{\frac{1}{2}|a+b|} = \frac{2(a\bullet b)^{\frac{1}{2}}}{|a+b|} \quad \text{(shown)}$$

Based on the diagram,

$$r = |v|\sin\theta \Rightarrow r^2 = |v|^2\sin^2\theta = |v|^2\left(1 - \cos^2\theta\right)$$

Substituting $|v| = (a\bullet b)^{\frac{1}{2}}$ and $\cos\theta = \dfrac{2(a\bullet b)^{\frac{1}{2}}}{|a+b|}$ into the above,

$$r^2 = (a\bullet b)\left[1 - \frac{4(a\bullet b)}{|a+b|^2}\right]$$

$$= (a\bullet b)\left[\frac{|a+b|^2 - 4a\bullet b}{|a+b|^2}\right]$$

$$= (a\bullet b)\left[\frac{(a+b)\bullet(a+b) - 4a\bullet b}{|a+b|^2}\right]$$

$$= (a\bullet b)\left[\frac{a\bullet a + 2a\bullet b + b\bullet b - 4a\bullet b}{|a+b|^2}\right]$$

$$= (a\bullet b)\left[\frac{a\bullet a - 2a\bullet b + b\bullet b}{|a+b|^2}\right]$$

$$= \left(a \bullet b\right) \left[\frac{(a - b) \bullet (a - b)}{|a + b|^2} \right]$$

$$= \left(a \bullet b\right) \left[\frac{|a - b|^2}{|a + b|^2} \right]$$

$$\therefore \quad |a + b|^2 \, r^2 = (a \bullet b) \, |a - b|^2 \quad \text{(shown)}$$