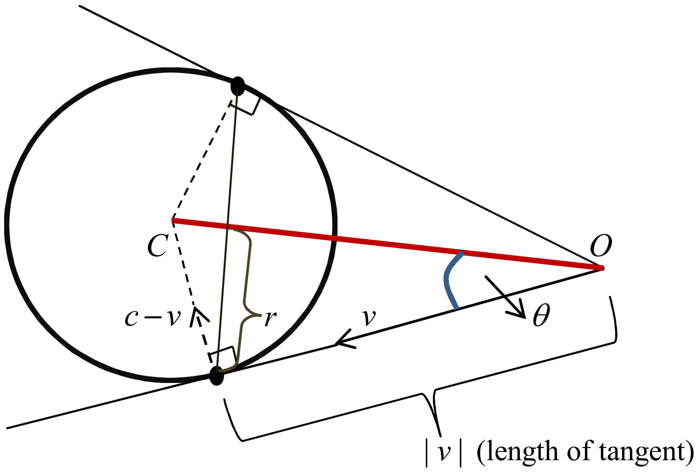
Extreme Problem 1 Solutions



Radius of sphere $=\dfrac{|b-a|}{2}$

Let position vector of centre of sphere be c, then $c=\dfrac{a+b}{2}$

Denoting the loci of a random point on the sphere as v,

Equation of the sphere is therefore given by $(v-c)\bullet(v-c)=\left[\dfrac{|b-a|}{2}\right]^2$

$$(v-c)\bullet(v-c)=\frac{1}{4}|b-a|^2=\frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v-2v\bullet c+c\bullet c=\frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v-2v\bullet c+\left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right)=\frac{1}{4}(b-a)\bullet(b-a)---------(1)$$

If tangents can be drawn from O to the sphere, then

$$v\bullet(c-v)=0\Rightarrow v\bullet c=v\bullet v$$

Substituting this into (1),

$$v\bullet v-2v\bullet v+\left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right)=\frac{1}{4}(b-a)\bullet(b-a)$$

$$v\bullet v=\left(\frac{a+b}{2}\right)\bullet\left(\frac{a+b}{2}\right)-\frac{1}{4}(b-a)\bullet(b-a)$$

$$|v|^2=\frac{1}{4}\left[(a+b)\bullet(a+b)-(b-a)\bullet(b-a)\right]$$

$$=\frac{1}{4}\left[(a\bullet a+2a\bullet b+b\bullet b)-(b\bullet b-2a\bullet b+a\bullet a)\right]$$

$$=\frac{1}{4}(4a\bullet b)=a\bullet b$$

$\therefore \quad |v|=(a\bullet b)^{\frac{1}{2}}$ (shown)

$v\bullet c=|v||c|\cos\theta$

Since $v\bullet c=v\bullet v$, $\cos\theta=\dfrac{v\bullet c}{|v||c|}=\dfrac{v\bullet v}{|v||c|}=\dfrac{|v|^2}{|v||c|}$

$$=\frac{|v|}{|c|}=\frac{(a\bullet b)^{\frac{1}{2}}}{\frac{1}{2}|a+b|}=\frac{2(a\bullet b)^{\frac{1}{2}}}{|a+b|} \text{ (shown)}$$

Based on the diagram,

$r=|v|\sin\theta\Rightarrow r^2=|v|^2\sin^2\theta=|v|^2\left(1-\cos^2\theta\right)$

Substituting $|v|=(a\bullet b)^{\frac{1}{2}}$ and $\cos\theta=\dfrac{2(a\bullet b)^{\frac{1}{2}}}{|a+b|}$ into the above,

$$r^2=(a\bullet b)\left[1-\frac{4(a\bullet b)}{|a+b|^2}\right]$$

$$=(a\bullet b)\left[\frac{|a+b|^2-4a\bullet b}{|a+b|^2}\right]$$

$$=(a\bullet b)\left[\frac{(a+b)\bullet(a+b)-4a\bullet b}{|a+b|^2}\right]$$

$$=(a\bullet b)\left[\frac{a\bullet a+2a\bullet b+b\bullet b-4a\bullet b}{|a+b|^2}\right]$$

$$=(a\bullet b)\left[\frac{a\bullet a-2a\bullet b+b\bullet b}{|a+b|^2}\right]$$

$$= (a \bullet b)\left[\frac{(a-b)\bullet(a-b)}{|a+b|^2}\right]$$

$$= (a \bullet b)\left[\frac{|a-b|^2}{|a+b|^2}\right]$$

$$\therefore \quad |a+b|^2\, r^2 = (a \bullet b)\,|a-b|^2 \quad \text{(shown)}$$