7. Integration and applications

-ability to adequately handle basic integration of algebraic polynomial structures, trigonometric functions(including inverse trigonometric functions), exponential functions and hybrid of two or more family of functions.

-ability to perform integration utilising techniques including **integration by parts**, **substitution** as well as evaluating definite integrals.

-ability to construct integrals to compute area of a specific region-this applies to both direct x-y functions as well as **parametric equations.**

-appreciate the concept of integration in the context of finding **volumes of revolution;** care must be taken to ensure that the integral is properly formulated for volume of revolution of shaded regions about a certain axis, **especially when it is other than the x or y axis.**

PREDICTED QUESTION STRUCTURES :

*a. (i) $\int \frac{dx}{\sqrt{x^2 - x + 1}}$ (Hint: Make use of the substitution $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\vartheta$)

(ii) $\int x\sin^{-1}(x^2)dx$

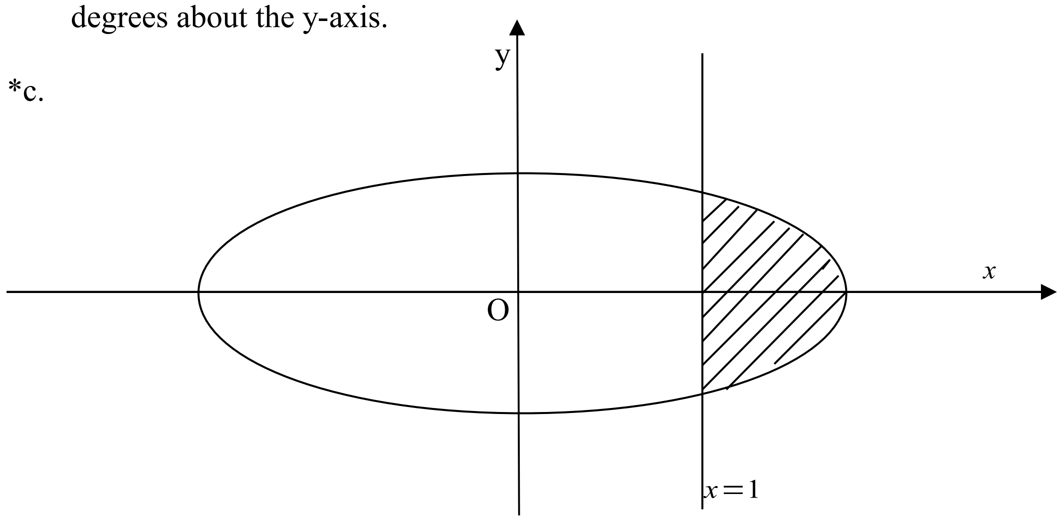
(iii) $\int \sin(\ln x)dx$

b. The region R is bounded by the curve $y = -\sqrt{25 - x^2}$ and the line $y = x - 5$

(i) Calculate the area of R.

(ii) The vertical line x=4 divides the region R into two regions where S is the larger region. Find the volume of the solid of revolution obtained when S is rotated 360 degrees about the y-axis.

*c.



The figure above shows the ellipse $(\frac{x}{2})^2 + y^2 = 1$ and the line x=1.

(i) By using the substitution $x = 2\sin\theta$, find the area of the shaded region, giving your answer in an exact form.

(ii) Calculate the volume generated by rotating the shaded region about the y-axis through 2π radians.

*d. An arc or the curve called the cycloid is given by the parametric equations $x = a(t - \sin t)$, $y = a(1 - \cos t)$ for $0 \le t \le 2\pi$. Show that the gradient of the tangent at the point P, with parameter t, is $\cot\frac{t}{2}$ and sketch the curve for $0 \le t \le 2\pi$. Show that the area enclosed by the $x-$ axis is $3\pi\left(a\right)^2$.

*e. With the aid of the GC, sketch the graph of $y = \dfrac{1}{4 + x^2}$ for $x \ge 0$. The region bounded by the curve $y = \dfrac{1}{4 + x^2}$ and the lines $x = 2$ and $y = \dfrac{1}{4}$ is denoted by R. Show that the volume of the solid obtained when R is rotated completely about the line $x = 2$ can be expressed in the form

$$\pi \ln 2 - 4\pi \int_{\frac{1}{8}}^{\frac{1}{4}} \sqrt{\frac{1 - 4y}{y}}\,dy$$

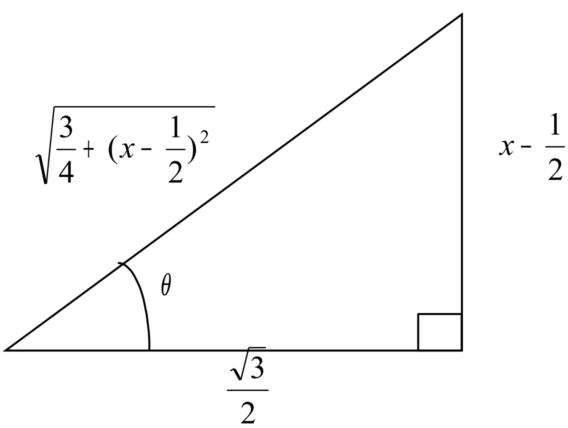
FULL SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

a (i) $$\int \frac{dx}{\sqrt{x^2 - x + 1}} = \int \frac{1}{\sqrt{(x - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}dx$$

Using the substitution $x - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\tan\theta$, the integral becomes

$$\int \frac{1}{\sqrt{\frac{3}{4}\tan^2\theta + \frac{3}{4}}}\left(\frac{\sqrt{3}}{2}\sec\theta\right)d\theta = \int \frac{1}{\frac{\sqrt{3}}{2}\sec\theta}\left(\frac{\sqrt{3}}{2}\sec\theta\right)d\theta = \int (\sec\theta)\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{2}{\sqrt{3}}\left[\sqrt{\frac{3}{4} + (x - \frac{1}{2})^2}\right] + \frac{2}{\sqrt{3}}(x - \frac{1}{2})\right| + C \text{ (shown)}$$



(ii) $\int x\sin^{-1}(x^2)dx = \frac{x^2}{2}(\sin^{-1}x^2) - \int \frac{x^2}{2}(\frac{1}{\sqrt{1-x^4}})(2x)dx$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \int (\frac{x^3}{\sqrt{1-x^4}})dx$$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{4}\int (\frac{4x^3}{\sqrt{1-x^4}})dx \quad = \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{4}(2)\sqrt{1-x^4} + C$$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{2}\sqrt{1-x^4} + C \text{ (shown)}$$

(iii) $\int \sin(\ln x)dx = x\sin(\ln x) - \int x(\frac{1}{x})\cos(\ln x)dx = x\ln(\sin x) - \int \cos(\ln x)dx$

$$= x\ln(\sin x) - [x\cos(\ln x) - \int (-x)(\frac{1}{x})\sin(\ln x)dx]$$

$$= x\ln(\sin x) - x\cos(\ln x) - \int \sin(\ln x)dx + B$$

Therefore, $2\int \sin(\ln x)dx = x\ln(\sin x) - x\cos(\ln x)$

$$\Rightarrow \int \sin(\ln x)dx = \frac{x}{2}[\ln(\sin x) - \cos(\ln x)] + A \text{ (shown)}$$

c.(i) Equation for upper part of ellipse is : $y = \sqrt{1-\left(\frac{x}{2}\right)^2}$

Area= $2\int_1^2 ydx = 2\int_1^2 \sqrt{1-\left(\frac{x}{2}\right)^2}dx = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,(2\cos\theta)d\theta$

$$=2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\cos^2\theta)d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta$$

$$=2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ sq units (shown)}$$

(ii) Volume= $2\left[\pi\int_0^{\frac{\sqrt{3}}{2}} x^2dy - \pi\left(\frac{\sqrt{3}}{2}\right)(1)^2\right] = 2\left[\pi\int_0^{\frac{\sqrt{3}}{2}} 4(1-y^2)dy - \pi\left(\frac{\sqrt{3}}{2}\right)\right]$

=10.88 cubic units (shown)

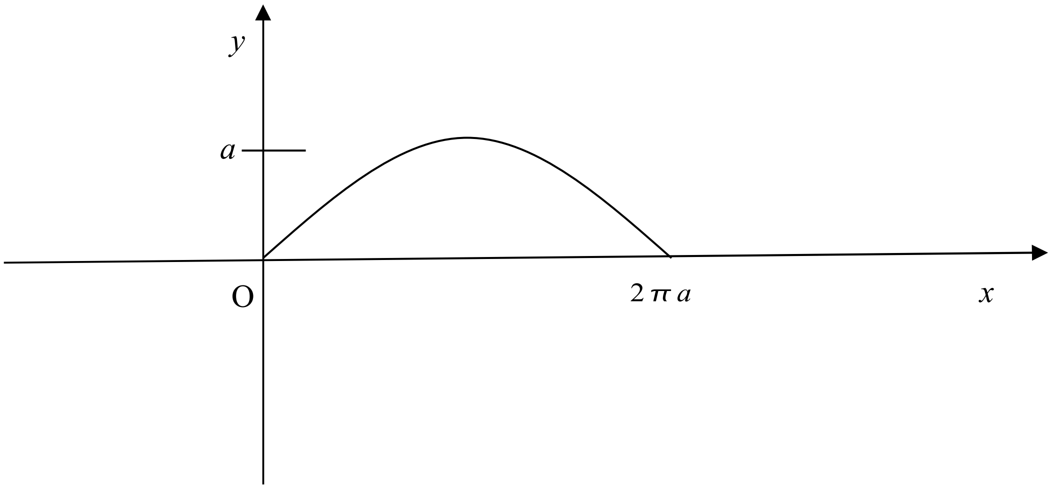
d. $x = a(t-\sin t) = at - a\sin t \Rightarrow \frac{dx}{dt} = a - a\cos t$

$y = a(1-\cos t) = a - a\cos t \Rightarrow \frac{dy}{dt} = a\sin t$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{a\sin t}{a-a\cos t}=\frac{a\sin t}{a(1-\cos t)}=\frac{a\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)}{a\left[1-(1-2\sin^2\frac{t}{2}\right]}$$

$$=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}=\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}=\cot\frac{t}{2}\ \text{(shown)}$$

(Note: the **double angle formulas** $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1-2\sin^2 x$ were used in the above formulation)



$$\text{Area} = \int_0^{2\pi a} y\,dx = \int_0^{2\pi a} y\frac{dx}{dt}dt = \int_0^{2\pi} a(1-\cos t)(a-a\cos t)dt$$

$$=\int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt$$

$$=a^2\int_0^{2\pi}(1-\cos t)^2\,dt$$

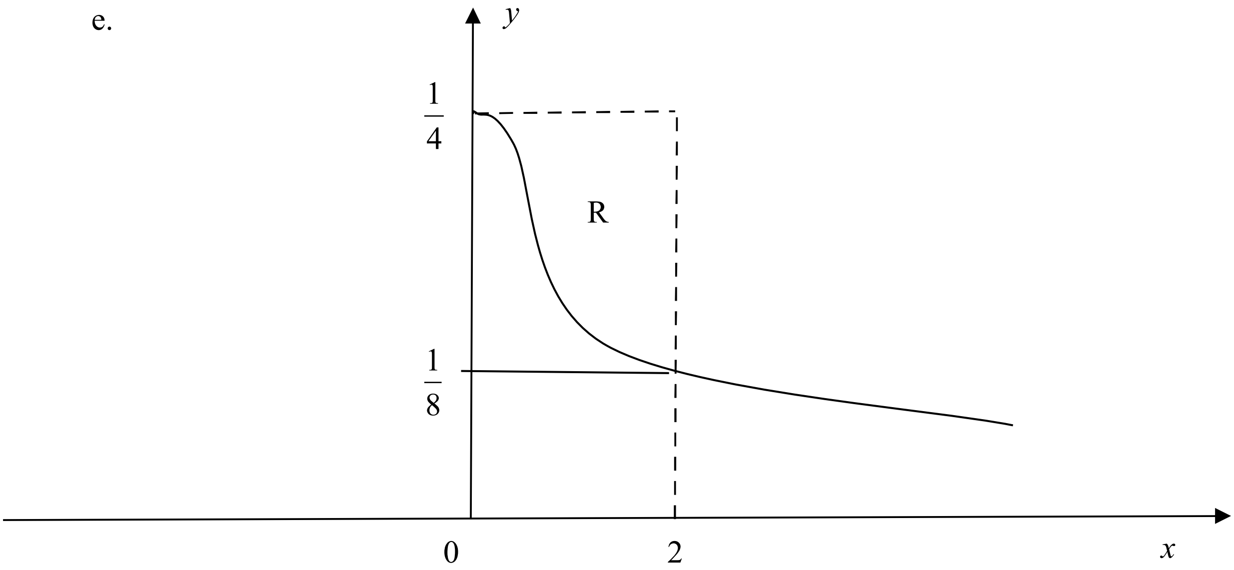
$$=a^2\int_0^{2\pi}\left(1-2\cos t+\cos^2 t\right)dt$$

$$=a^2\int_0^{2\pi}\left(1-2\cos t+\frac{\cos 2t+1}{2}\right)dt$$

$$=a^2\left[t-2\sin t+\frac{1}{4}\sin 2t+\frac{1}{2}t\right]_0^{2\pi}$$

$$=a^2\left[(2\pi+\pi)-(0)\right]=3\pi\,(a)^2\ \text{(shown)}$$

e.



Volume generated $V = \pi \int_{\frac{1}{8}}^{\frac{1}{4}} (2 - x)^2 dy$ -----------------(1)

$y = \frac{1}{4 + x^2} \Rightarrow 4 + x^2 = \frac{1}{y}$

$x^2 = \frac{1}{y} - 4 = \frac{1 - 4y}{y}$

$x = \sqrt{\frac{1 - 4y}{y}}$

Put this into (1), we have $V = \pi \int_{\frac{1}{8}}^{\frac{1}{4}} (2 - x)^2 dy = \pi \int_{\frac{1}{8}}^{\frac{1}{4}} 4 - 4x + x^2 dy$

$= \pi \int_{\frac{1}{8}}^{\frac{1}{4}} 4 - 4\sqrt{\frac{1 - 4y}{y}} + \frac{1 - 4y}{y} dy$

$= \pi \int_{\frac{1}{8}}^{\frac{1}{4}} 4 - 4\sqrt{\frac{1 - 4y}{y}} + \frac{1}{y} - 4 dy$

$= \pi \int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{y} - 4\sqrt{\frac{1 - 4y}{y}} dy$

$$=\left[\ln\left(\left(\frac{1}{4}\right)\div\left(\frac{1}{8}\right)\right)\right]\pi - 4\pi\int_{\frac{1}{8}}^{\frac{1}{4}}\sqrt{\frac{1-4y}{y}}\,dy$$

$$=\pi\ \ln 2 - 4\pi\int_{\frac{1}{8}}^{\frac{1}{4}}\sqrt{\frac{1-4y}{y}}\,dy \text{ (shown)}$$

8. Complex numbers

-ability to relate to the core formulas of complex numbers, ie $z+z^*=2\text{Re}(z)$, $zz^*=|z|^2$, $\arg(z_1 z_2)=\arg(z_1)+\arg(z_2)$, $(a+b)^*=a^*+b^*$ etc ; these are **highly relevant** especially in the context of **abstract complex number** questions.

-possess a firm grasp of the various complex number representations, namely **cartesian, polar and exponential**(or Euler's) forms.

-appreciate the **geometrical implications** of multiplying (or dividing) one complex number by another (eg multiplying a complex number by $2e^{i(\frac{\pi}{4})}$ has the effect of scaling the original complex number's magnitude by a factor of 2 and increasing its argument by $\frac{\pi}{4}$ in the Argand diagram). Note that questions which test such concepts typically involve regular shapes such as squares or triangles **where one or more vertices** are to be solved for.

-ability to solve for roots of unity and **presenting the roots of a polynomial equation in terms of paired quadratic factors** (Note this is extended to solving n^{th} degree polynomials, eg $z^6=-64$).

-appreciate the concept of **loci** in the complex number context and able to sketch the loci of **circles, perpendicular bisectors and half lines**, and subsequently solving for various points of intersection, maximum and minimum distances between fixed and variable points based on **geometrical and trigonometrical observations.**

PREDICTED QUESTION STRUCTURES :

*a. . Given that $w = -2-2\sqrt{3}i$, express w^n in the polar form, where n is an integer. Find $w^n + (w^n)^*$ in terms of n. Hence deduce the value of $w^n + (w^n)^*$ in terms of n, if n is not a multiple of 3.

b. Express $a = \frac{7+4i}{3-2i}$ in the form x+iy where x and y are real.
Sketch in an Argand diagram the locus of point P representing the complex number z such that $|z+a|=|a|$. Hence find the greatest value of $|z+1|$.

*c. Given that z is a complex number, show that $|iz-1+i| = |z+1+i|$.

Sketch the following loci in a single Argand diagram, labeling each locus clearly.
(i) |z+1+i|=|1-i|

(ii) arg (iz-1+i)= $-\dfrac{3\pi}{4}$

Find the complex number that represents the point of intersection of the two loci.

*d. By using a graphical method, or otherwise, find the complex numbers z such that |z-2|=$\sqrt{11}$ and |z-3|=4.

*e. . By letting $z = \cos\theta + i\sin\theta = e^{i\theta}$, show that $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$. By considering the expansion of $(z+\dfrac{1}{z})^4$, show that $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$. Find a similar expression for $\sin^4\theta$ by considering $(z-\dfrac{1}{z})^4$.

*f. (i) Show that, for all complex numbers z and all real numbers α,
$(z-e^{i\alpha})(z-e^{-i\alpha}) = z^2 - 2(\cos\alpha)z + 1$.
(ii) Find the seven complex numbers which satisfy the equation $z^7 - 1 = 0$.

(iii) Hence, show that, for all complex numbers z,
$z^7 - 1 = (z-1)[z^2 - 2\cos(\dfrac{2\pi}{7})z + 1][z^2 - 2\cos(\dfrac{4\pi}{7})z + 1][z^2 - 2\cos(\dfrac{6\pi}{7})z + 1]$.

FULL SOLUTIONS FOR QUESTIONS MARKED WITH ASTERIX:

a. $w = -2 - 2\sqrt{3}i = 4[\cos(-\dfrac{2\pi}{3}) + i\sin(-\dfrac{2\pi}{3})]$

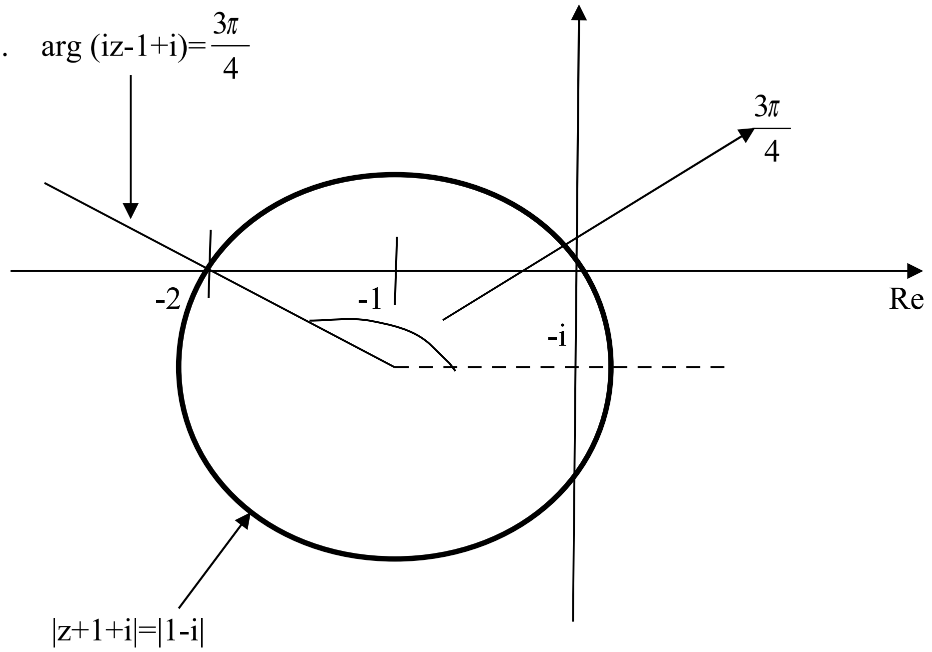
$\therefore\ w^n = 4^n[\cos(-\dfrac{2n\pi}{3}) + i\sin(-\dfrac{2n\pi}{3})]$ (shown)

$w^n + (w^n)^* = 4^n[\cos(-\dfrac{2n\pi}{3}) + i\sin(-\dfrac{2n\pi}{3})] + 4^n[\cos(-\dfrac{2n\pi}{3}) - i\sin(-\dfrac{2n\pi}{3})]$

$= 2(4^n)\cos(-\dfrac{2n\pi}{3})$

$n \neq 3k \Rightarrow w^n + (w^n)^* = 2(4^n)(-0.5) = -4^n$ (shown) $[\cos(\dfrac{2n\pi}{3}) = -0.5]$.

c.



(i) $|z+1+i|=|1-i| \Rightarrow |z+1+i|=\sqrt{2}$ (circle with centre -1-i and radius of $\sqrt{2}$ units)

(ii) $iz-1+i=(i)(z-\frac{1}{i}+1)=(i)(z+i+1)$

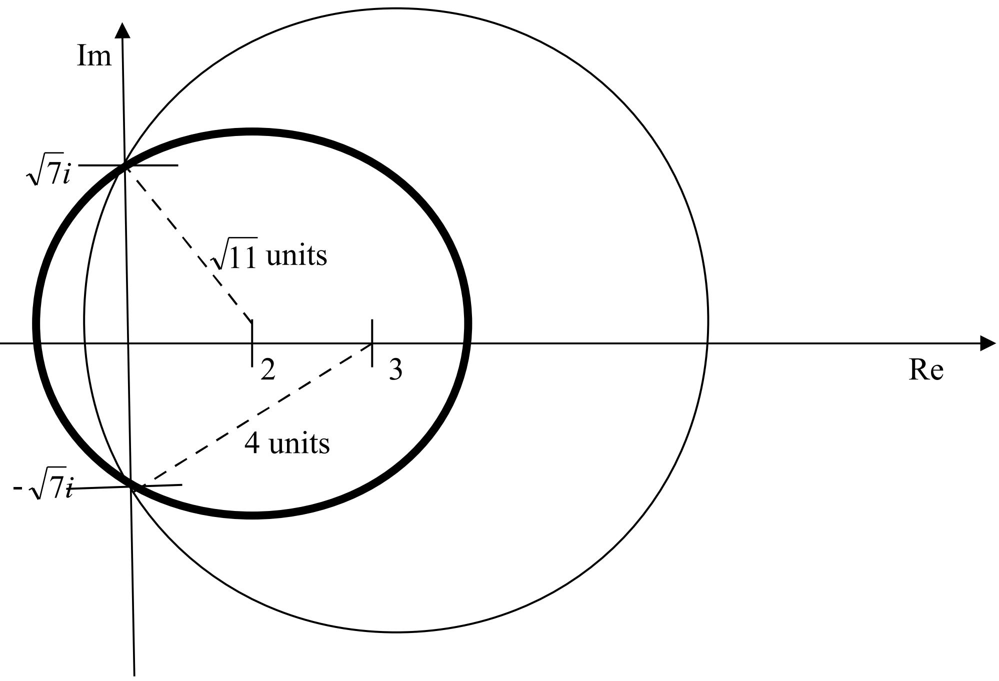
$\therefore$ $\arg(iz-1+i)=\arg[(i)(z+i+1)]=\arg(i)+\arg(z+i+1)=\frac{\pi}{2}+\arg(z+i+1)$

Hence, $\arg(iz-1+i)=-\frac{3\pi}{4} \Rightarrow \frac{\pi}{2}+\arg(z+i+1)=-\frac{3\pi}{4}$,

ie $\arg(z+i+1)=-\frac{5\pi}{4}=\frac{3\pi}{4}$ (line pivoted at -1-i and inclined at an angle of $\frac{3\pi}{4}$ to the horizontal)

From the argand diagram, point of intersection is z= -2. (shown)

d. .



From the Argand diagram, point of intersection is $\pm\sqrt{7}i$. (shown)

e. $z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$z^{-n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

$\therefore\ z^n + z^{-n} = 2\cos n\theta \qquad z^n - z^{-n} = 2i\sin n\theta$

$$(z+\frac{1}{z})^4 = z^4 + 4z^2 + 6 + 4(\frac{1}{z^2}) + \frac{1}{z^4} = (z^4 + \frac{1}{z^4}) + 6 + 4(z^2 + \frac{1}{z^2})$$

$\Rightarrow\ (2\cos\theta)^4 = 2\cos 4\theta + 6 + 8\cos 2\theta$

$16\cos^4\theta = 2\cos 4\theta + 6 + 8\cos 2\theta$

$\therefore\ 8\cos^4\theta = \cos 4\theta + 3 + 4\cos 2\theta$ (shown)

$$(z-\frac{1}{z})^4 = z^4 - 4z^2 + 6 - 4(\frac{1}{z^2}) + \frac{1}{z^4} = (z^4 + \frac{1}{z^4}) + 6 - 4(z^2 + \frac{1}{z^2})$$

$\Rightarrow\ (2i\sin\theta)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$

$16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$ (shown)

f. (i) $\left(z - e^{i\alpha}\right)\left(z - e^{-i\alpha}\right) = z^2 - z\left(e^{i\alpha} + e^{-i\alpha}\right) + 1$

$= z^2 - 2z\cos\alpha + 1$ (shown)

(ii) $z^7 - 1 = 0 \Rightarrow z^7 = 1 = e^{i2k\pi}$

$\therefore\ z = e^{(i)\left(\frac{2k\pi}{7}\right)}, \quad k = 0 \to 7$

(iii) $$z^7 - 1 = (z-1)\left(z - e^{(i)\left(\frac{2\pi}{7}\right)}\right)\left(z - e^{-(i)\left(\frac{2\pi}{7}\right)}\right)\left(z - e^{(i)\left(\frac{4\pi}{7}\right)}\right)\left(z - e^{-(i)\left(\frac{4\pi}{7}\right)}\right)$$

$$\left(z - e^{(i)\left(\frac{6\pi}{7}\right)}\right)\left(z - e^{-(i)\left(\frac{6\pi}{7}\right)}\right)$$

$$=(z-1)\left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right) \text{ (shown)}$$

9. Vectors

-ability to understand the core vector concepts namely: ratio theorem, **projection vectors** (including **foot of perpendicular** from point to plane/line, shortest distance from point to plane/line), **cartesian, parametric and scalar product** representation of planes and lines, **acute angles** between **lines**, **a plane and a line** as well as **between 2 planes**, **intersection of 2 lines** and **intersection of 2 or 3 planes**(note that one popular feature of questions is to disguise the solving of intesection of planes under the pretext of asking the student to intepret set of conditions for a system of linear equations), **skew lines.**

-appreciate the notion of **scalar and vector(cross) products**, and their application in solving abstract vector questions (Note that the cross product is also commonly employed in **finding area of parallelograms as and triangles**)

-ability to reconcile the various concepts coherently when solving questions and competently understand the unique configuration of the question context to enhance efficiency.

PREDICTED QUESTION STRUCTURES :

a. ABC is a triangle in which $A\hat{C}B = 90^\circ$. The position vectors of A and B relative to the origin are 2**i**-5**j**+6**k** and 3**i**-10**j**+6**k** respectively and the Cartesian equations of BC are $x - 3 = -\frac{y}{4} - \frac{5}{2} = \frac{z}{2} - 3$. Show that a possible vector equation of BC is **r**=3**i**-10**j**+6**k**+λ (**i**-4**j**+2**k**). where λ is a scalar. Find

(i) the coordinates of C,
(ii) The area of the triangle ABC,
(iii) The distance from C to AB.

*b. A line L whose Cartesian equation is $\frac{x}{2} + 1 = y = -(z - 6)$ meets the plane Π with Cartesian equation -2x+3y+z=6 at the point P. The point Q with position vector 4**i**+3**j**-3**k** lies on the line L.

(i) Write down a vector equation of the line L and the vector equation of the plane Π in scalar product form.

(ii) Find the position vector of P.

(iii) Show that the acute angle between line L and plane Π is $\sin^{-1}(\frac{1}{\sqrt{21}})$. Hence, or otherwise, find the length of the projection of PQ onto the plane Π .

*c. Find, in the form $r \bullet n = d$, the equation of the plane containing three points whose position vectors are –**i**+**j**+**k**, **i**+2**j**+2**k** and 2**i**+2**j**+4**k**.
Find, in the form **r**=**p**+t**q**, the equation of the image of the line **r**=-**i**+**j**+**k**+λ (**i**-5**k**) reflected in this plane.

*d. Relative to an origin O, the position vectors of points A and B are **a** and **b** respectively, and **c** is the position vector of the point C on AB which divides AB in the ratio 3: 1. Given that angle AOB is acute, show that the length d of the projection of $\overrightarrow{OC}$ on $\overrightarrow{OB}$ is given by

$$d=\frac{3}{4}|b|+\frac{a \bullet b}{4|b|}.$$

*e. Given that the system of linear equations

$$x+\alpha y+3z=\beta \quad (\alpha, \beta \in R)$$
$$3x+y+4z=9$$
$$x+y=3$$

has infinite solutions, obtain the numerical values of α and β.

FULL SOLUTIONS TO QUESTIONS MARKED WITH ASTERIX:

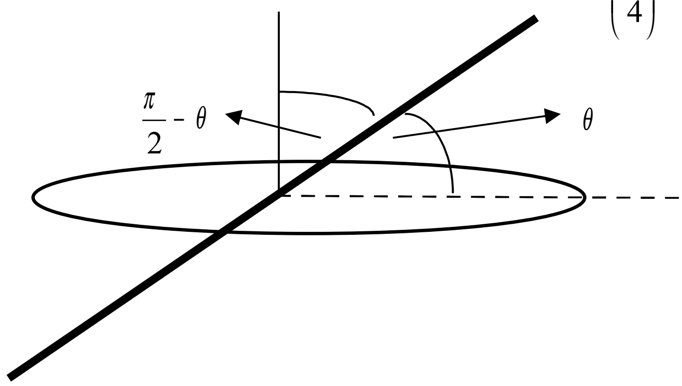
b. (i) $\frac{x+2}{2} = y = \frac{z-6}{-1} \Rightarrow$ equation of line L is $r = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

and equation of plane π is $r \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 6$ (shown)

(ii) When L intersects π at P, $\begin{pmatrix} -2+2\lambda \\ \lambda \\ 6-\lambda \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 6$

Solving gives $\lambda = 2$. Hence, position vector of P=$\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$. (shown)

(iii)



$$\left|\begin{pmatrix}-2\\3\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}\right| = \sqrt{14}\sqrt{6}\cos(\frac{\pi}{2}-\theta) = \sqrt{14}\sqrt{6}\sin\theta$$

Hence, $\sin\theta = \frac{2}{\sqrt{84}} = \frac{1}{\sqrt{21}} \Rightarrow \theta = \sin^{-1}(\frac{1}{\sqrt{21}})$ (shown)

c. $n = [\begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}-1\\1\\1\end{pmatrix}] \times [\begin{pmatrix}2\\2\\4\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix}] = \begin{pmatrix}2\\-3\\-1\end{pmatrix}$

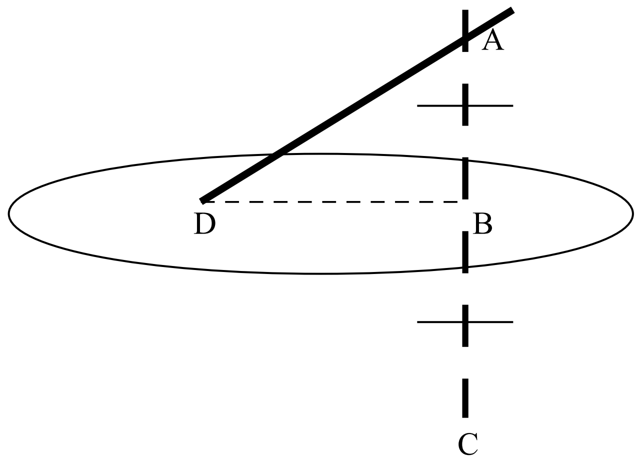
Therefore, equation of plane is $r\bullet\begin{pmatrix}2\\-3\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\-1\end{pmatrix} = -6$

$$\Rightarrow r\bullet\begin{pmatrix}-2\\3\\1\end{pmatrix} = 6 \quad \text{(shown)}$$

$$\begin{pmatrix}-2\\3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-5\end{pmatrix} \neq 0 \Rightarrow$$

line to be reflected is not parallel to the plane and cuts the plane at an angle.

Choosing point A by assigning $\lambda = 1 \Rightarrow \overrightarrow{OA} = \begin{pmatrix}0\\1\\-4\end{pmatrix}$



Equation of line AB is $r = \begin{pmatrix}0\\1\\-4\end{pmatrix} + t\begin{pmatrix}-2\\3\\1\end{pmatrix}$

When this line intersects the plane, $\begin{pmatrix} -2t \\ 1+3t \\ -4+t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 6 \Rightarrow \ t = \dfrac{1}{2}$

$$\therefore \ \overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \dfrac{1}{2}\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}] - \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

$$\overrightarrow{AC} = 2\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

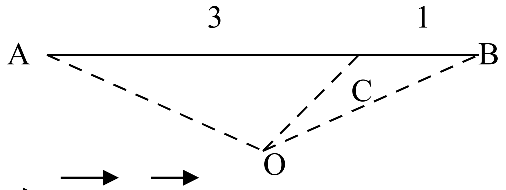
$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$ parallel to $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

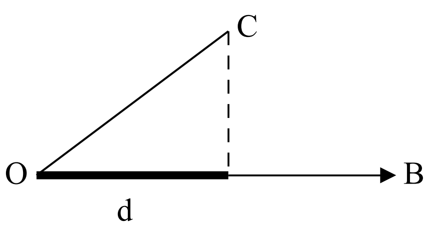
Hence, equation of mirror image of line **reflected** in plane is

$$r = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \quad \text{(shown)}$$

d.



$$\overrightarrow{OC} = \frac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4} = \frac{1}{4}a + \frac{3}{4}b$$



$$d = \left|OC \bullet \hat{b}\right| = \left|\left(\frac{a}{4}+\frac{3b}{4}\right)\bullet \hat{b}\right| = \left|\left(\frac{a}{4}+\frac{3b}{4}\right)\bullet b\right| \left|\frac{1}{b}\right| = \left|\frac{a\bullet b}{4}+\frac{3b\bullet b}{4}\right| \frac{1}{|b|}$$

$$= \left|\frac{a\bullet b}{4|b|}+\frac{3|b|^2}{4|b|}\right| = \frac{a\bullet b}{4|b|}+\frac{3|b|^2}{4|b|} \quad [\ a \bullet b > 0 \text{ since angle AOB is acute}]$$

$$= \frac{3|b|}{4}+\frac{a\bullet b}{4|b|} \text{ (shown)}$$

e. The question can be interpreted in the context of 3 planes having a common line of intersection.

Considering the 2 planes with equations $3x+y+4z=9$ and $x+y=9$, their common line of intersection can be found as follows :

$3x+y+4z=9$ ----------------(1)

$x+y=3$ ----------------------(2)

From (2), we have $y=3-x$; substituting this into (1) gives

$3x+3-x+4z=9 \Rightarrow 2x+4z=6 \Rightarrow z=\frac{3}{2}-\frac{x}{2}$

Hence, line of intersection L is given by $r=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\3-x\\\frac{3}{2}-\frac{x}{2}\end{pmatrix}=\begin{pmatrix}0\\3\\\frac{3}{2}\end{pmatrix}+x\begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix}$

$$=\begin{pmatrix}0\\3\\\frac{3}{2}\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

(Note that x is merely a dummy parameter)

In addition, line L MUST also lie on the plane with equation $x+\alpha y+3z=\beta$, ie

$\begin{pmatrix}2\\-2\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\\alpha\\3\end{pmatrix}=0 \Rightarrow \alpha=-\frac{1}{2}$ (L is **perpendicular** to the normal of the third plane containing unknowns α and β.)

and $\begin{pmatrix}0\\3\\\frac{3}{2}\end{pmatrix}\bullet\begin{pmatrix}1\\\alpha\\3\end{pmatrix}=\beta \Rightarrow \beta=3\alpha+\frac{9}{2}=3(-\frac{1}{2})+\frac{9}{2}=3$ (shown)

10. Inequalities/ System of linear equations

- ability to solve inequalities either through **mathematical means** or **graphical methods** (as required by the question); be thorough and careful when manipulating/transferring terms across the inequality signage (eg dividing/ multiplying both sides by a negative term results in a directional change of the inequality, while appending the natural logarithm to both sides does not yield any change whatsoever)

- ability to recognise the significance of proving an expression to **be strictly positive (usually through completing the square)** in helping to expedite the inequality solving process.

-ability to **properly dismantle the modulus** encapsulating certain functions (eg for $|x+1|>|2x-3|$, squaring both sides serve to eliminate the modulus brackets, ie $(x+1)^2>(2x-3)^2$.)

-ability to **modify original solution set to deduce** the solution set for extended versions/variations of the basic inequality previously examined in in the question.

-ability to **decipher and derive a set of linear equations** (typically 3) based on the context of the question-the storyline varies greatly and students must be astute enough to **realise the motivation of the question.** (Popular structures include a polynomial curve threading through a set of given points, where the student is required to obtain the equation of the polynomial itself, and business schemes which investigate the concept of profit margins related to the sale of different products).

PREDICTED QUESTION STRUCTURES :

*a. Solve the inequality $\frac{1}{x-3}<\frac{x}{x+1}$, giving your answers in exact form.

Hence, solve the inequality $\frac{1}{e^{-x}-3}<\frac{1}{1+e^{x}}$.

b. Using the method of completing the square, show that $4x^2-4x+3$ is always positive.

Hence, without the use of a graphic calculator, solve the inequality $\frac{32x-243}{x^2+7x-60}>4$.

*c. A, B and C denote angles which constitute a triangle. It is further known that: $\sin(A-B)=\frac{1}{2}$ and $\sin(B-C)=\cos(B-C)$. Find the values of A, B and C in degrees.

FULL SOLUTIONS TO QUESTIONS MARKED WITH ASTERIX:

a. $\frac{1}{x-3}<\frac{x}{x+1}$

$$\frac{1}{x-3} - \frac{x}{x+1} < 0$$

$$\frac{x+1-x(x-3)}{(x-3)(x+1)} < 0$$

+ve	-ve	+ve	-ve	+ve
-1	$2-\sqrt{5}$	3	$2+\sqrt{5}$	

$$\frac{-x^2+4x+1}{(x-3)(x+1)} < 0$$

$$\frac{x^2-4x-1}{(x-3)(x+1)} > 0$$

$$\frac{(x-2)^2-5}{(x-3)(x+1)} > 0$$

$\therefore \quad x < -1, \; 2-\sqrt{5} < x < 3, \quad x > 2+\sqrt{5}$ (shown)

$$\frac{1}{e^{-x}-3} < \frac{1}{1+e^{x}}$$

$$\frac{1}{e^{-x}-3} < \frac{1}{1+\frac{1}{e^{-x}}}$$

$$\frac{1}{e^{-x}-3} < \frac{e^{-x}}{e^{-x}+1}$$

$\therefore \quad 0 < e^{-x} < 3 \Rightarrow \; x > -\ln 3$

and $e^{-x} > 2+\sqrt{5} \Rightarrow \; x < -\ln(2+\sqrt{5})$ (shown)

c. $A+B+C=180$ ------------------(1)

Since $\sin(A-B)=\frac{1}{2}$, then $A-B=\sin^{-1}(\frac{1}{2})=30$ ----------------(2)

$\sin(B-C)=\cos(B-C)\Rightarrow\tan(B-C)=1\Rightarrow B-C=\tan^{-1}(1)=45$ ------------------(3)

Combining the 3 equations to form the augmented matrix gives

$$\begin{matrix} 1 & 1 & 1 & 180 \\ 1 & -1 & 0 & 30 \\ 0 & 1 & -1 & 45 \end{matrix}$$

Solving the matrix system by the graphic calculator gives

$A=95^o, B=65^o$ and $C=20^o$ (shown)