<u>**Integration Techniques Summary**</u>

1. List of basic formulas:

Function	**Integrated Result**
ax^n	$\frac{ax^{n+1}}{n+1}$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{(a)(n+1)}$
$[f'(x)][f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1}$
$\frac{[f'(x)]}{a^2+[f(x)]^2}$	$\frac{1}{a}\tan^{-1}\left[\frac{f(x)}{a}\right]$
$\frac{[f'(x)]}{[f(x)]^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{f(x)-a}{f(x)+a}\right\|$
$\frac{[f'(x)]}{a^2-[f(x)]^2}$	$\frac{1}{2a}\ln\left\|\frac{a+f(x)}{a-f(x)}\right\|$
$\frac{[f'(x)]}{\sqrt{a^2-[f(x)]^2}}$	$\sin^{-1}\left[\frac{f(x)}{a}\right]$
$f'(x)a^{f(x)}$	$\frac{1}{\ln a}a^{f(x)}$
$f'(x)e^{f(x)}$	$e^{f(x)}$
$\frac{f'(x)}{f(x)}$	$\ln\|f(x)\|$
Function	**Integrated Result**

$f'(x)\cos[f(x)]$	$\sin[f(x)]$
$f'(x)\sin[f(x)]$	$-\cos[f(x)]$
$f'(x)\sec^2[f(x)]$	$\tan[f(x)]$
$f'(x)\cos ec^2[f(x)]$	$-\cot[f(x)]$
$f'(x)\cos ec[f(x)]\cot[f(x)]$	$-\cos ec[f(x)]$
$\tan x$	$-\ln\vert\cos x\vert$
$\cot x$	$\ln\vert\sin x\vert$
$\sec x$	$\ln\vert\sec x+\tan x\vert$
$\cos ecx$	$\ln\vert\cos ecx-\cot x\vert$

2. Miscellaneous trigonometric integrals:

a. $\int \sin^n xdx$ and $\int \cos^n xdx$ **:**

If n is **odd**, separate a single $\sin x$ or $\cos x$ from the original function, and use the identity $\sin^2 x+\cos^2 x=1$.

Example: $\int \sin^3 xdx=\int(\sin^2 x)(\sin x)dx=\int(1-\cos^2 x)(\sin x)dx$

$$=\int \sin x-\cos^2 x\sin xdx=-\cos x+\frac{\cos^3 x}{3}+C \text{ (shown)}$$

If n is **even**, use the double angle formula of either $\cos 2x=2\cos^2 x-1$ or $\cos 2x=1-2\sin^2 x$ for conversion.

Example: $\int \cos^2 xdx=\int \frac{\cos 2x+1}{2}dx=\frac{1}{4}\sin 2x+\frac{1}{2}x+C$ (shown)

b. $\int \tan^n xdx$:

Separate $\tan^2 x$ from the original function, and use the identity $1+\tan^2 x=\sec^2 x$.

Example: $\int \tan^5 x dx = \int (\tan^2 x)(\tan^3 x) dx = \int (\sec^2 x - 1)(\tan^3 x) dx$

$$= \int \sec^2 x \tan^3 x - \tan^3 x dx$$

$$= \int \sec^2 x \tan^3 x - \tan^2 x (\tan x) dx$$

$$= \int \sec^2 x \tan^3 x - (\sec^2 x - 1)(\tan x) dx$$

$$= \int \sec^2 x \tan^3 x - \sec^2 x \tan x + \tan x dx$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C \text{ (shown)}$$

c. $\int \sin(mx)\sin(nx)dx$ or $\int \sin(mx)\cos(nx)dx$ or $\int \cos(mx)\cos(nx)dx$:

Use one of the 3 identities below to transform the product to a sum or difference:

(i) $\sin(mx)\cos(nx) = \frac{1}{2}\{\sin[(m+n)x] + \sin[(m-n)x]\}$

(ii) $\sin(mx)\sin(nx) = \frac{1}{2}\{\cos[(m-n)x] - \cos[(m+n)x]\}$

(iii) $\cos(mx)\cos(nx) = \frac{1}{2}\{\cos[(m-n)x] + \cos[(m+n)x]\}$

Example: $\int \cos 3x \cos x dx = \frac{1}{2}\int \cos 4x + \cos 2x dx = \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + C$ (shown)

3. Integration via substitution:

Generally, to find an integral by means of a substitution $x = f(u)$,

(i) Differentiate x wrt u to arrive at $\frac{dx}{du} = f'(u) \Rightarrow dx = f'(u)du$

(ii) Subsequently replace all x by $f(u)$ and dx by $f'(u)du$

(iii) Perform the integration and remember to convert the result back to x for an indefinite integral (**not needed** for definite integrals)

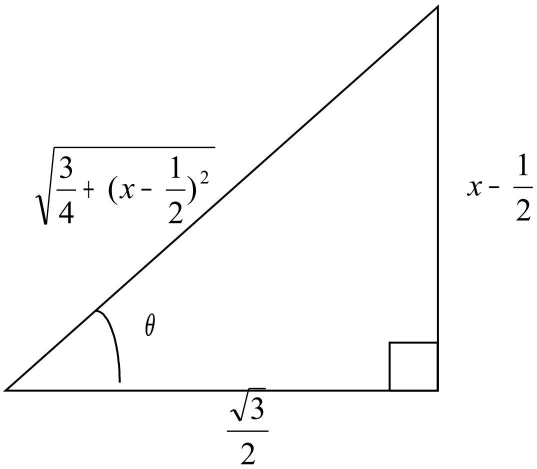
Example: Evaluate $\int \frac{dx}{\sqrt{x^2 - x + 1}}$ by considering the substitution $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$.

$$\int \frac{dx}{\sqrt{x^2 - x + 1}} = \int \frac{1}{\sqrt{(x - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}} dx$$

Using the substitution $x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, the integral becomes

$$\int \frac{1}{\sqrt{\frac{3}{4}\tan^2\theta+\frac{3}{4}}}\left(\frac{\sqrt{3}}{2}\sec\theta\right)d\theta = \int \frac{1}{\frac{\sqrt{3}}{2}\sec\theta}\left(\frac{\sqrt{3}}{2}\sec\theta\right)d\theta = \int (\sec\theta)\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{2}{\sqrt{3}}\left[\sqrt{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}\right] + \frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right| + C \text{ (shown)}$$



4. Integration by parts:

> $\int uv'\,dx = uv - \int u'v\,dx$ where u is a function which can be differentiated and v is a function that can be easily reduced via integration. Note that integration by parts is only feasible if out of the product of two functions, **at least one is directly integrable.**

Example:

$$\int x\sin^{-1}(x^2)\,dx = \frac{x^2}{2}(\sin^{-1}x^2) - \int \frac{x^2}{2}\left(\frac{1}{\sqrt{1-x^4}}\right)(2x)\,dx$$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \int \left(\frac{x^3}{\sqrt{1-x^4}}\right)dx$$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{4}\int \left(\frac{4x^3}{\sqrt{1-x^4}}\right)dx = \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{4}(2)\sqrt{1-x^4} + C$$

$$= \frac{x^2}{2}(\sin^{-1}x^2) - \frac{1}{2}\sqrt{1-x^4} + C \text{ (shown)}$$

5. **Integrals of the form** $\int \frac{cx+d}{ax^2+bx+c}dx$

> (i) Rewrite $\int \frac{cx+d}{ax^2+bx+c}dx$ as $\int \frac{P(2ax+b)+Q}{ax^2+bx+c} = \int \frac{P(2ax+b)}{ax^2+bx+c} + \frac{Q}{ax^2+bx+c}dx$,
> where $2ax+b$ is clearly the derivative of ax^2+bx+c.
>
> (ii) $\int \frac{P(2ax+b)}{ax^2+bx+c}$ gives $P\ln|ax^2+bx+c|$

(iii) $\int \frac{Q}{ax^2+bx+c}dx$ can be easily integrated via completing the square method for the denominator.

Example: $\int \frac{x-1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C \text{ (shown)}$$

6. Integrals of the form $\int \frac{cx^2+dx+e}{ax^2+bx+c}dx$

(i) Rewrite $\int \frac{cx^2+dx+e}{ax^2+bx+c}dx = \int \frac{P(ax^2+bx+c)+Q(2ax+b)+R}{ax^2+bx+c}dx$

$$= \int P + \frac{Q(2ax+b)}{ax^2+bx+c} + \frac{R}{ax^2+bx+c}dx$$

$$= Px + \int \frac{Q(2ax+b)}{ax^2+bx+c} + \frac{R}{ax^2+bx+c}dx$$

where $2ax+b$ is clearly the derivative of ax^2+bx+c.

(ii) $\int \frac{Q(2ax+b)}{ax^2+bx+c}$ gives $Q\ln|ax^2+bx+c|$

(iii) $\int \frac{R}{ax^2+bx+c}dx$ can be easily integrated via completing the square method for the denominator.

Example: $\int \frac{2x^2+4x+1}{x^2+x+1}dx = \int \frac{2(x^2+x+1)+(2x+1)-2}{x^2+x+1}dx$

$$= 2x + \int \frac{2x+1}{x^2+x+1} - \frac{2}{x^2+x+1}dx$$

$$= 2x + \ln|x^2 + x + 1| - 2\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$= 2x + \ln|x^2 + x + 1| - 2\left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\right]\tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= 2x + \ln|x^2 + x + 1| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C \text{ (shown)}$$