The idea to evaluate integrals involving modulus is as such:

For example, considering $\int_a^c |f(x)|\ dx$, we must first know what range of values of x for which $f(x)$ is negative, in this case lets assume it to be $f(x) \le 0$ for $a \le x \le b$, where $b < c$.

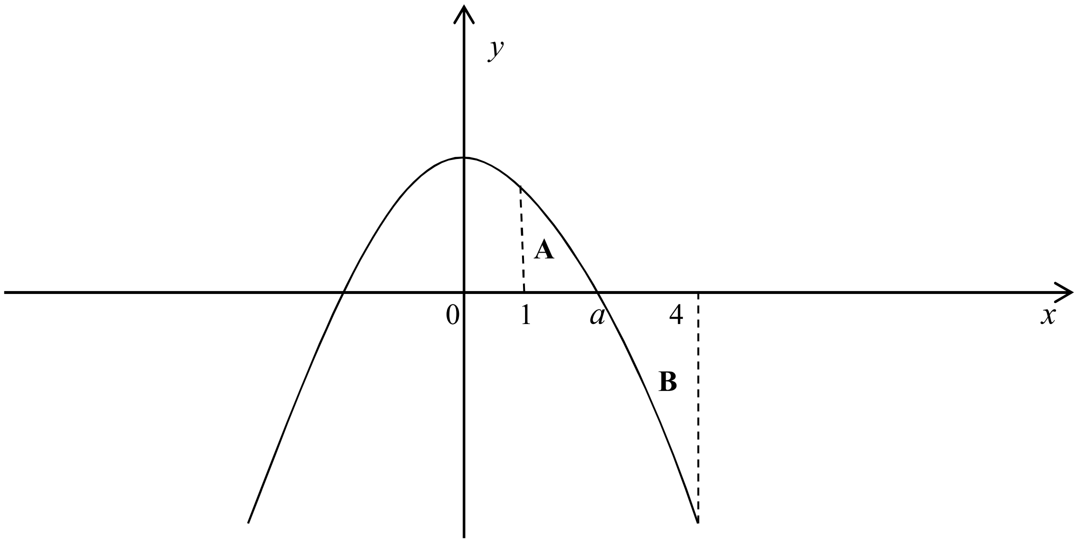
Then $\int_a^c |f(x)|\ dx = -\int_a^b f(x)\ dx + \int_b^c f(x)\ dx$

We section the integral into two parts, appending an additional negative sign to the integral part for which $f(x) \le 0$, while simply integrating the remaining second part normally without the modulus sign.

With that, examine the following two problems:

(a) Solving $\int_1^4 |a^2 - x^2|\ dx$

For $y = a^2 - x^2$, where $1 < a < 4$, the graph is given below:



Noting that $a^2 - x^2 \ge 0$ for $1 \le x \le a$ and $a^2 - x^2 \le 0$ for $a \le x \le 4$,

$$\int_1^4 |a^2 - x^2|\ dx = \int_1^a a^2 - x^2\ dx - \int_a^4 a^2 - x^2\ dx$$

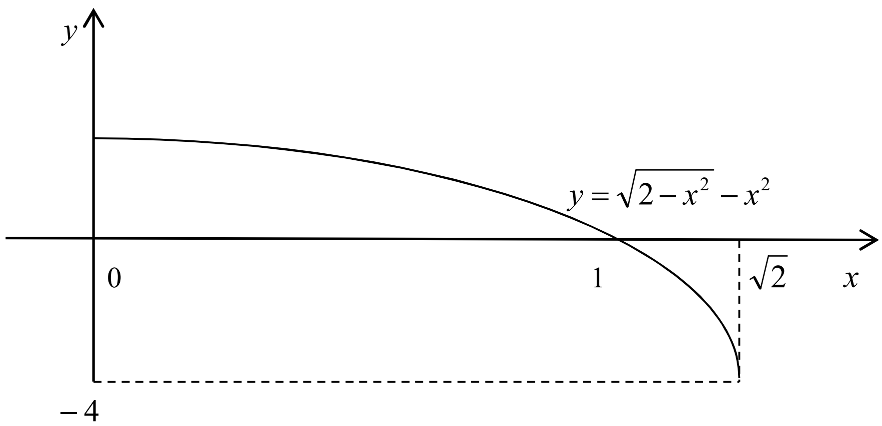
$$= \left[a^2 x - \frac{x^3}{3}\right]_1^a - \left[a^2 x - \frac{x^3}{3}\right]_a^4 = \left(\frac{2a^3}{3} - a^2 + \frac{1}{3}\right) - \left(4a^2 - \frac{64}{3} - \frac{2a^3}{3}\right)$$

$$= \frac{4a^3}{3} - 5a^2 + \frac{65}{3} \quad \text{(shown)}$$

$\int_1^4 \left| a^2 - x^2 \right| dx$ can also be interpreted as **finding the sum of the areas** of regions A and B marked on the above graph.

(b) Solving $\int_0^{\sqrt{2}} \left| \sqrt{2 - x^2} - x^2 \right| dx$

For $y = \sqrt{2 - x^2} - x^2$, the graph looks like this for $0 \le x \le \sqrt{2}$:



Hence, $\int_0^{\sqrt{2}} \left| \sqrt{2 - x^2} - x^2 \right| dx = \int_0^1 \sqrt{2 - x^2} - x^2 \; dx - \int_1^{\sqrt{2}} \sqrt{2 - x^2} - x^2 \; dx$

$$= 0.9521 - (-0.3241) = 1.2762 \quad \text{(shown)}$$

(Note: The above definite integrals were obtained using the GC, but if you wish to manually integrate them to obtain exact values, to handle the $\sqrt{2 - x^2}$ component, you will have to use the substitution $x = \sqrt{2}\sin\theta$.)