Differential Equations Summary

1. First order differential equations

a. Variables Separable DE:

Arrange through manipulation such that the form below is achieved:

$$f(x)dx = g(y)dy$$

Integrate subsequently to yield the required solution.

Example: Solve $\frac{dy}{dx} = 1 - y$ for y<1.

SOLUTION :

$$\frac{dy}{dx} = 1 - y \Rightarrow \frac{1}{1-y}\frac{dy}{dx} = 1$$

$$-\int \frac{-1}{1-y}dy = \int dx$$

$$-\ln|1-y| = x + C$$

Since $y < 1$, $-\ln(1-y) = x + C$

$$1 - y = e^{-x+B}$$

$\therefore$ $y = 1 - Ae^{-x}$ where $A = e^{B}$, $B = -c$ (shown)

This solution is commonly termed the **GENERAL SOLUTION**, where A is unknown. When initial conditions are provided, eg y =0 when x =0, then A assumes a specific value and the solution is termed the **PARTICULAR SOLUTION**. When we use the GC to plot out a series of graphs for various values of A, the result is that we produce **a family of solution curves**.

b. Reduction through substitution:

The introduction of an intermediate variable **aids in reducing the original differential equation to a far simpler version** which is readily solvable.

Example: Use the substitution y= vx, where v is a function of x, to solve the differential equation $x\frac{dy}{dx} = 3x + y$.

SOLUTION :

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into the differential equation gives

$$x\left(v + x\frac{dv}{dx}\right) = 3x + vx$$

$$x^2 \frac{dv}{dx} = 3x \Rightarrow \frac{dv}{dx} = \frac{3}{x}$$

$$\int dv = \int \frac{3}{x} dx$$

$$\therefore \ v = 3\ln|x| + C$$

$$y = 3x\ln|x| + Cx \quad \text{(shown)}$$

2. Second order differential equations:

Second order DEs are typically of the form $\frac{d^2y}{dx^2} = f(x)$, whereby running the DE through two iterations of integration will yield the required solution.

Example: $\frac{d^2s}{dt^2} = -g$

Integrating twice $wrt \ \ t$ gives: $\frac{ds}{dt} = -gt + A$

and $s = -\frac{gt^2}{2} + At + B$ (shown)

3. Modelling a problem through the usage of a differential equation:

Typically the question demands the student to first formulate a DE relating to the context of the situation, and subsequently solve it. Realise that the formulation of a DE involves the following considerations:

(i) Constants of proportionality

(ii) Net rate, which is usually composed of an **"in" rate and "out" rate,** eg birth rate - death rate

Example: The growth of a particular species of insect is studied in an experimental environment. The rate of death is proportional to the number in thousands, x, of insects, at any time t days after the start of the experiment. The rate of birth is proportional to x^2. When $x = 2$, the number of larvae hatched is equal to the number of insects that died. Show that $\frac{dx}{dt} = ax(x-2)$ where a is a constant.

SOLUTION :

$\frac{dx}{dt} =$ Birth rate minus death rate

$= ax^2 - bx$

When $x = 2, \frac{dx}{dt} = 0$

$a(2)^2 - 2b = 0 \Rightarrow b = 2a$

$\therefore \ \dfrac{dx}{dt} = ax^2 - 2ax = ax(x-2)$ (shown)

Example: A rectangular tank has a horizontal base. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds the depth of the water in the tank is x metres. If the depth is 0.5m, it remains at this constant value. Show that $\dfrac{dx}{dt} = -k(2x-1)$, where k is a positive constant.

<u>SOLUTION :</u>

$\dfrac{dx}{dt} =$ flowing in rate minus flowing out rate

$= k - Ax$

When $x = 0.5, \dfrac{dx}{dt} = 0$

$k - \dfrac{1}{2}A = 0 \Rightarrow A = 2k$

$\therefore \ \dfrac{dx}{dt} = k - 2kx = -2kx + k = -k(2x-1)$ (shown)