Binomial Series Summary

1. Expansion of $\left[1+f(x)\right]^n$

This is given by

$$1+nf(x)+\frac{(n)(n-1)}{2!}\left[f(x)\right]^2+\frac{(n)(n-1)(n-2)}{3!}\left[f(x)\right]^3+\frac{(n)(n-1)(n-2)(n-3)}{4!}\left[f(x)\right]^4+\ldots\ldots\ldots\ldots$$

Example: $(1-3x)^{-2}=1+(-3x)(-2)+\frac{(-2)(-3)}{2!}(-3x)^2+\frac{(-2)(-3)(-4)}{3!}(-3x)^3+\ldots\ldots\ldots\ldots$

$$=1+6x+27x^2+108x^3+\ldots\ldots\ldots\ldots$$

Note:

(i) For $n\in\Re,\ n\neq Z^+$, the structure intended for expansion **must strictly adhere to the form** $\left[1+f(x)\right]^n$. For instance, $(8+2x)^{\frac{1}{3}}$ must be reorganized as $8^{\frac{1}{3}}\left(1+\frac{x}{4}\right)^{\frac{1}{3}}$ at the very beginning before proceeding with the actual binomial series expansion.

(ii) The expansion of $\left[1+f(x)\right]^n$ **is valid** for $|f(x)|<1.$

2. Typical applications of the binomial series expansion

(a) To provide an estimation of specific surd values:

Example: $(8-x)^{\frac{1}{3}}\approx 2-\frac{1}{12}x+\frac{1}{288}x^2$; by substituting $x=\frac{1}{8}$, we can arrive at a value for $\sqrt[3]{63}$:

$$\left(\frac{63}{8}\right)^{\frac{1}{3}}=2-\frac{1}{12}\left(\frac{1}{8}\right)+\frac{1}{288}\left(\frac{1}{64}\right)=2-\frac{1}{96}+\frac{1}{18432}$$

$$\frac{1}{2}\left(\sqrt[3]{63}\right)=\frac{36673}{18432}$$

$$\text{Hence, } \sqrt[3]{63}=\frac{36673}{9216}\ \text{(shown)}$$

(b) To compute the approximate equations of tangents to points on a sophisticated curve:

Example: if we are aware that $y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}\approx 1-\frac{1}{2}x+\frac{1}{8}x^2$,

Then $\frac{dy}{dx}\approx\frac{d}{dx}\left(1-\frac{1}{2}x+\frac{1}{8}x^2\right)=-\frac{1}{2}+\frac{1}{4}x$

Hence, at the point $\left(\frac{1}{3},\frac{1}{\sqrt[4]{2}}\right)$, gradient of the tangent to the curve $y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}$ is approximately equal to $-\frac{1}{2}+\frac{1}{4}\left(\frac{1}{3}\right)=-\frac{5}{12}$, and the equation of this tangent is simply given by

$y-\frac{1}{\sqrt[4]{2}}=-\frac{5}{12}\left(x-\frac{1}{3}\right)$, ie $y=-\frac{5}{12}x+\left(\frac{5}{36}+\frac{1}{\sqrt[4]{2}}\right)$ (shown)

3. Finding the formula of the general term in the expansion

Example: In the expansion of $(x^2+4)^{-2}=\frac{1}{16}\left(1+\frac{x^2}{4}\right)^{-2}$, coefficient of x^{2n}

$$=\left(\frac{1}{16}\right)\frac{(-2)(-3)(-4)\text{.........}(-n-1)}{n!}\left(\frac{1}{4}\right)^n=\left(\frac{1}{16}\right)\frac{(-1)^n(2)(3)(4)\text{........}(n+1)}{n!}\left(\frac{1}{4}\right)^n$$

$$=\left(\frac{1}{16}\right)\frac{(-1)^n(n+1)!}{n!}\left(\frac{1}{4}\right)^n=\frac{(-1)^n(n+1)}{4^n(16)}=\frac{(-1)^n(n+1)}{4^{n+2}}\text{ (shown)}$$

4. Ascending series expansion

Example: $\left(1+\frac{1}{4x^2}\right)^{-\frac{1}{2}}$ may resemble the required format $[1+f(x)]^n$, but if an ascending series is to be achieved through expansion, then $f(x)=\frac{1}{4x^2}$ will not be appropriate as the main variable x resides purely in the denominator. A transformation has to be made:

$$\left(1+\frac{1}{4x^2}\right)^{-\frac{1}{2}}=\left(\frac{1}{4x^2}\right)^{-\frac{1}{2}}\left(4x^2+1\right)^{-\frac{1}{2}}$$

$$=(2x)\left[1+\left(-\frac{1}{2}\right)(4x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(4x^2)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(4x^2)^3+.......\right]$$