# Applications of Differentiation Summary

## 1. Parametric Equations/Tangents/Normals:

For a set of parametric descriptions whereby $x = f(t),\ \ y = g(t)$,

$$\boxed{\frac{dy}{dx} = \frac{g'(t)}{f'(t)}}$$

Equation of **tangent**: $y - g(t) = \frac{g'(t)}{f'(t)}\left[x - f(t)\right]$

Equation of **normal**: $y - g(t) = -\frac{f'(t)}{g'(t)}\left[x - f(t)\right]$

**Question types:**

Example: A curve has parametric equations

$$x = 1 + 2\sin\theta, \quad y = 4 + \cos\theta.$$

P is a point on the curve where $\theta = \frac{\pi}{6}$. Find the area of the triangle bounded by the tangent and normal at P, as well as the y-axis.

**SOLUTIONS :**

At point $P$, $x = 2,\ \ y = 4 + \frac{\sqrt{3}}{2}$

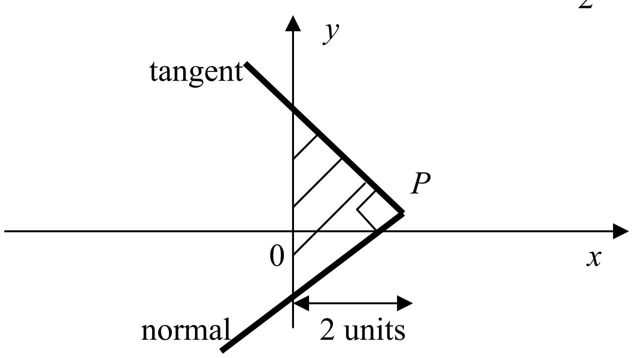
$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta \Rightarrow$ At P, $\frac{dy}{dx} = -\frac{1}{2\sqrt{3}}$

Equation of tangent: $y - (4 + \frac{\sqrt{3}}{2}) = -\frac{1}{2\sqrt{3}}(x - 2)$

When it cuts the $y-axis$, $x = 0 \Rightarrow y = 4 + \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}$

Equation of normal: $y - (4 + \frac{\sqrt{3}}{2}) = 2\sqrt{3}(x - 2)$

When it cuts the $y-axis$, $x = 0 \Rightarrow y = 4 - \frac{7\sqrt{3}}{2}$



Area of triangle = $\frac{1}{2}(2)\left[\left(4+\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\right)-\left(4-\frac{7\sqrt{3}}{2}\right)\right]=4\sqrt{3}+\frac{1}{\sqrt{3}}=\frac{13}{\sqrt{3}}$ units

(shown)

Example: A curve is defined by the parametric equations $x=t^2, y=t^3$. Show that the equation of the tangent to the curve at the point P ($p^2, p^3$) is $2y-3px+p^3=0$. Show that there is just one point on the curve at which the tangent passes through the point (-3,-5), and determine the coordinates of this point.

**SOLUTIONS :**

$\frac{dy}{dx}=\frac{3t^2}{2t}=\frac{3t}{2}$

At $P$, $\frac{dy}{dx}=\frac{3}{2}p$

Equation of tangent is $y-p^3=\frac{3}{2}p\ (x-p^2)$

Rearranging gives $2y-3px+p^3=0$ (shown)

Since it passes through (-3,-5),

2(-5)-3 $p$ (-3)+ $p^3$=0 $\Rightarrow -10+9p+p^3=0$

$(p-1)(p^2+p+10)=0$

$\therefore\ p=1$ and point is $(1^2,\ 1^3)=(1, 1)$ (shown)

(Quadratic polynomial has no real roots since determinant $=b^2-4ac=-39<0$)

2. **Rates of Change:**

The chain rule is typically used in such questions; variations of this rule include the following:

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt},\quad \frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt},\quad \frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt} \text{ etc}$$

Where $V$, $A$, $r$, $h$ typically denote volume, area, radius and height quantities respectively. Ensure **knowledge of the separate derivatives are thorough and clear**, eg $\frac{dV}{dt}$ describes the rate of change of volume.

**Popular formulas:**

Volume of Cone $-\frac{1}{3}\pi(r^2)h$ ; Volume of Sphere $-\frac{4}{3}\pi(r^3)$;

Surface area of Sphere $-4\pi(r^2)$

**Question types:**

Example: An inverted cone of base radius 4cm and height 8cm is initially filled with water. Water drips out from the vertex at a rate of $2\pi \ cm^3 s^{-1}$. Find the rate of decrease in the depth of the water in the cone 16 seconds after the dripping starts.

**SOLUTIONS :**

$$\frac{r}{h}=\frac{4}{8}=\frac{1}{2}\Rightarrow r=\frac{1}{2}h$$

At any time t, volume remaining V=$\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi(\frac{1}{2}h)^2(h)=\frac{1}{12}\pi h^3$

Differentiating V with respect to $h$, $\frac{dV}{dh}=\frac{\pi}{4}h^2$

When t =16, $\frac{1}{12}\pi h^3=\frac{1}{3}\pi(4)^2(8)-2(16)\Rightarrow h^3=128$

By the chain rule, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}\Rightarrow -2\pi=\frac{\pi}{4}(\sqrt[3]{128})^2\times\frac{dh}{dt}$

$$\therefore \frac{dh}{dt}=-0.315cms^{-1} \text{ (shown)}$$

**3. Maxima/Minima problems:**

Maxima/minima problems typically employ the concept of **stationary values**, and ascertaining the nature of those values (ie maximum or minimum) via the **sign test** or **investigating the second derivative** if it is reasonably easy to attain.

**Question types:**

Example:
A length of channel of given depth d is to be made from a rectangular sheet of metal of width 2a. The metal is to be bent in such a way that the cross section ABCD is as shown in the figure, with AB+BC+CD= $2a$ and with AB and CD inclined to the line BC at an angle $\theta$.


Show that BC=2( $a-d\operatorname{cosec}\theta$) and that the area of the cross section ABCD is $2ad+d^2(\cot\theta-2\operatorname{cosec}\theta)$.

Show that the maximum value of $2ad+d^2(\cot\theta-2\operatorname{cosec}\theta)$, as $\theta$ varies, is $d(2a-d\sqrt{3})$.

By considering the length of BC, show that the cross sectional area can only be made equal to this maximum value if $2d \le a\sqrt{3}$.

**SOLUTIONS :**

$$\sin\theta = \frac{d}{AB} \Rightarrow AB = CD = d\operatorname{cosec}\theta$$

$BC = 2a - AB - CD = 2a - 2d\operatorname{cosec}\theta = 2(a - d\operatorname{cosec}\theta)$ (shown)

Area of cross section $ABCD$

$= 2d(a-d\operatorname{cosec}\theta) + 2\left[\frac{1}{2}(d)(d\cot\theta)\right] = 2d(a-d\operatorname{cosec}\theta) + d(d\cot\theta)$

$= 2ad + d^2(\cot\theta - 2\operatorname{cosec}\theta)$ (shown)

Let $y = 2ad + d^2(\cot\theta - 2\operatorname{cosec}\theta)$

$$\frac{dy}{d\theta} = d^2\left[-\operatorname{cosec}^2\theta + 2\cot\theta\operatorname{cosec}\theta\right]$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \operatorname{cosec}^2\theta = 2\cot\theta\operatorname{cosec}\theta \to \operatorname{cosec}\theta = 2\cot\theta$$

$$\therefore\ \cos\theta = \frac{1}{2},\quad \theta = \frac{\pi}{3}$$

Maximum value $= 2ad + d^2\left[\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right] = d(2a - d\sqrt{3})$ (shown)

$BC = 2(a - d\operatorname{cosec}\theta)$, putting in $\theta = \frac{\pi}{3}$,

$$BC = 2\left(a - \frac{2}{\sqrt{3}}d\right)$$

Since $BC \ge 0$, $a \ge \frac{2}{\sqrt{3}}d \Rightarrow 2d \le \sqrt{3}a$ (shown)

Example:

The equation of a curve is $y = ax^2 - 2bx + c$, where a, b and c are constants, a>0. Find, in terms of a, b and c, the coordinates of the turning point on the curve, and state whether it is a maximum or minimum point.

**SOLUTIONS :**

$$y = ax^2 - 2bx + c \to \frac{dy}{dx} = 2ax - 2b$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{b}{a},\ y = c - \frac{b^2}{a}$$

$\frac{d^2y}{dx^2} = 2a > 0\ [\because\ a > 0]$ Hence, $\left(\frac{b}{a}, c - \frac{b^2}{a}\right)$ is a minimum point.