Additional Vectors Problems

1(i) A plane Π_1 has vector equation $r \bullet (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$.

(a) Find the coordinates of the foot of perpendicular from the origin to the plane.

(b) Determine whether the line l with equation $r = 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ lies in the plane.

(ii) A second plane Π_2 contains the line l and is perpendicular to Π_1. Find the equation of Π_2 in the form $r \bullet n = d$.

(iii) A third plane Π_3 is perpendicular to both planes Π_1 and Π_2, and is at a perpendicular distance of 3 units from the point (0, 2, 1). Find the coordinates of the possible points of intersection of the three planes.

2. Given that the two planes P_1 and P_2 have equations

$$r \bullet \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 1 \quad \text{and} \quad r \bullet \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = 5 \text{ respectively,}$$

(a) Verify that the line of intersection of these two planes has equation $r = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -22 \\ 42 \end{pmatrix}$.

(b) Find the equation of the plane through A (1, 3, -7) and which is perpendicular to both planes P_1 and P_2.

(c) Find the vector equation of the line through A which bisects the acute angle between the two planes P_1 and P_2.

(d) Using the results obtained in (a) and (c), find the equation of the plane which bisects the acute angle between the two planes P_1 and P_2.

3. The points A, B, C, which form the base of a tetrahedron, have position vectors a, b and c respectively. Vertex D has the position vector d. Find in terms of a, b and c,

(i) The area of the triangle ABC

(ii) a vector perpendicular to ABC

(iii) Show that the volume of the tetrahedron $ABCD$ is $\frac{1}{6}\left|d \bullet (b \times c + a \times b + c \times a) - a \bullet (b \times c)\right|$.

(The volume of a tetrahedron is equals to $\frac{1}{3} \times$ base area $\times$ perpendicular height.)

4. The planes $\prod_1$ and $\prod_2$, which meet in the line l, have vector equations

$$r = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_1 (2\mathbf{i} + 3\mathbf{k}) + \phi_1 (-4\mathbf{j} + 5\mathbf{k}),$$

$$r = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_2 (3\mathbf{j} + \mathbf{k}) + \phi_2 (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

respectively. Find a vector equation of the line l in the form $r = a + tb$.

Find the vector equation of the plane $\prod_3$ which contains the line l and passes through the point $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, giving it in Cartesian form. Deduce that the system of equations

$$6x - 5y - 4z = -32$$

$$5x - y + 3z = 24$$

$$9x - 2y + 5z = 40$$

possess an infinite number of solutions.

5. The lines l_1 and l_2 have equations

$$l_1 : r = (1 + 2\mu)\,\mathbf{i} + 2\mu\,\mathbf{j} - (4 + 3\mu)\,\mathbf{k}, \qquad \mu \in \Re$$

$$l_2 : r = (4 + a\lambda)\,\mathbf{i} + (6 + 4\lambda)\,\mathbf{j} + (2 + 9\lambda)\,\mathbf{k}, \qquad \lambda \in \Re$$

respectively, where a is a constant.

(a) Find, correct to the nearest degree, the acute angle between l_1 and the $x-$axis.

(b) The point A has position vector $2\mathbf{i} - 2\mathbf{j} + b\,\mathbf{k}$. Given that the line l_2 passes through the point A, find the values of a and b.

Hence find

(i) the position vector of the point of intersection between lines l_1 and l_2,

(ii) the position vector of the foot of perpendicular from point A to the line l_1,

(iii) the vector equation of the line l_3, the reflection of l_2 about l_1.

6. Relative to an origin O, points A and B have position vectors $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}$ respectively. The line l has vector equation $r=(6\mathbf{i}+a\mathbf{j})+t(\mathbf{i}+3\mathbf{j}+a\mathbf{k})$, where t is a real parameter and a is a constant. The line m passes through the point A and is parallel to the line OB.

(i) Find the position vector of the point P on m such that OP is perpendicular to m.

(ii) Show that the two lines l and m have no common point.

(iii) If the acute angle between the line l and the $z-$axis is 60°, find the exact values of the constant a.

7. By expanding $(b-c)\bullet(b-c)$, simplify $|b|^2+|c|^2-(b-c)\bullet(b-c)$.

Taking $\overrightarrow{AC}=b$ and $\overrightarrow{AB}=c$, deduce the cosine formula for triangle ABC.