Additional Revision Questions 4 Solutions

1. At the end of the first year, amount in bank $=1000(1.04)$

   At the end of the second year, amount in bank

   $=[1000(1.04)-50](1.04)=1000(1.04)^2-50(1.04)$

   At the end of the third year, amount in bank

   $=[1000(1.04)^2-50(1.04)-50](1.04)=1000(1.04)^3-50(1.04)^2-50(1.04)$

   Extending similar logic for the fourth year onwards,

   At the end of the nth year, amount in bank

   $=1000(1.04)^n-50(1.04)^{n-1}-50(1.04)^{n-2}-\dots\dots\dots-50(1.04)$

   $=1000(1.04)^n-50\left[1.04+1.04^2+1.04^3+\dots\dots+1.04^{n-1}\right]$

   $=1000(1.04)^n-50\left[\frac{1.04(1.04^{n-1}-1)}{1.04-1}\right]=1000(1.04)^n-1250(1.04)(1.04^{n-1}-1)$

   $=1000(1.04)^n-1250(1.04)^n+1300=1300-250(1.04)^n$

   Since \$50 shall be withdrawn at the beginning of the next year, amount in bank then

   $=1300-250(1.04)^n-50=1250-250(1.04)^n$

   $=1250\left[1-\frac{1}{5}(1.04)^n\right]$ (shown)

2(i) $\sum_{r=n+1}^{2n}(r-2n)^2=\sum_{r=n+1}^{2n}r^2-4nr+4n^2$

$$=\sum_{r=n+1}^{2n}r^2-4n\sum_{r=n+1}^{2n}r+\sum_{r=n+1}^{2n}4n^2$$

$$=\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)-4n\left[\frac{2n-(n+1)+1}{2}\right](n+1+2n)+[2n-(n+1)+1](4n^2)$$

$$=\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)-4n\left(\frac{n}{2}\right)(3n+1)+4n^3$$

$$= \frac{2n}{6}(2n+1)(4n+1) - \frac{n}{6}(n+1)(2n+1) - 2n^2(3n+1) + 4n^3$$

$$= \frac{n}{3}(2n+1)(4n+1) - \frac{n}{6}(n+1)(2n+1) - 2n^2(3n+1) + 4n^3$$

$$= \frac{n}{6}\left[2(2n+1)(4n+1) - (n+1)(2n+1) - 12n(3n+1) + 24n^2\right]$$

$$= \frac{n}{6}\left[16n^2 + 12n + 2 - 2n^2 - 3n - 1 - 36n^2 - 12n + 24n^2\right]$$

$$= \frac{n}{6}\left[2n^2 - 3n + 1\right] = \frac{n}{6}(2n-1)(n-1) \qquad \text{(shown)}$$

(ii) $\sum_{r=12}^{n}\left(2^r - 2^{r-1}\right) = \sum_{r=12}^{n} 2^r - \sum_{r=12}^{n} 2^{r-1}$

$$= \left(2^{12} + 2^{13} + \ldots\ldots\ldots + 2^{n-1} + 2^n\right) - \left(2^{11} + 2^{12} + \ldots\ldots\ldots + 2^{n-1}\right)$$

$$= 2^n - 2^{11} \quad \text{(shown)}$$

3. $\frac{r^2 + r - 1}{(r+1)!} = \frac{r^2 + r}{(r+1)!} - \frac{1}{(r+1)!}$

$$= \frac{r(r+1)}{(r+1)!} - \frac{1}{(r+1)!}$$

$$= \frac{r}{r!} - \frac{1}{(r+1)!}$$

$$= \frac{1}{(r-1)!} - \frac{1}{(r+1)!} \quad \text{(shown)}$$

$$\sum_{r=0}^{n} \frac{r^2 + r - 1}{(r+1)!} = -1 + \sum_{r=1}^{n} \frac{r^2 + r - 1}{(r+1)!}$$

$$= -1 + \sum_{r=1}^{n} \frac{1}{(r-1)!} - \frac{1}{(r+1)!}$$

$$= -1 + \left\{ \begin{array}{l} 1 - \frac{1}{2!} \\ 1 - \frac{1}{3!} \\ \frac{1}{2!} - \frac{1}{4!} \\ \frac{1}{3!} - \frac{1}{5!} \\ \quad \vdots \\ \frac{1}{(n-2)!} - \frac{1}{n!} \\ \frac{1}{(n-1)!} - \frac{1}{(n+1)!} \end{array} \right.$$

$$= -1 + 2 - \frac{1}{n!} - \frac{1}{(n+1)!}$$

$$= 1 - \frac{1}{n!} - \frac{1}{(n+1)!} = 1 - \left[\frac{1}{n!} + \frac{1}{(n+1)!} \right]$$

$$= 1 - \left[\frac{n+1}{(n+1)!} + \frac{1}{(n+1)!} \right]$$

$$= 1 - \frac{n+2}{(n+1)!} \text{ (shown)}$$

4(i) $\sum_{r=0}^{n} \frac{1}{3^r} \sin^3 \left(3^r \theta \right) = \sum_{r=0}^{n} \frac{1}{3^r} \bullet \frac{1}{4} \left[3\sin\left(3^r \theta\right) - \sin 3\left(3^r \theta\right) \right]$

$$= \frac{1}{4} \sum_{r=0}^{n} \frac{1}{3^r} \left[3\sin\left(3^r \theta\right) - \sin\left(3^{r+1} \theta\right) \right]$$

$$= \frac{1}{4} \left\{ \begin{array}{l} 3\sin\theta - \sin 3\theta \\ \sin 3\theta - \frac{1}{3}\sin 9\theta \\ \frac{1}{3}\sin 9\theta - \frac{1}{9}\sin 27\theta \\ \quad \vdots \\ \frac{1}{3^{n-1}} \sin\left(3^n \theta\right) - \frac{1}{3^n} \sin\left(3^{n+1} \theta\right) \end{array} \right.$$

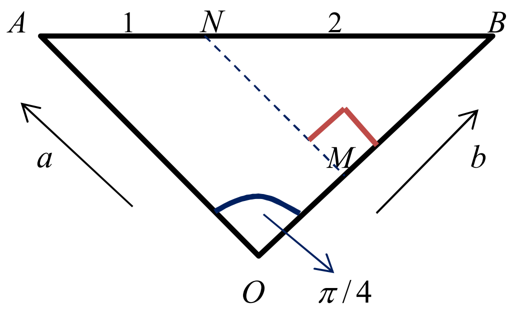
$$= \frac{1}{4} \left[3\sin\theta - \frac{1}{3^n} \sin\left(3^{n+1} \theta\right) \right] \text{ (shown)}$$

(ii) $\sin^3\left(\frac{\pi}{2}\right)+\frac{1}{3}\sin^3\left(\frac{3\pi}{2}\right)+\frac{1}{3^2}\sin^3\left(\frac{3^2\pi}{2}\right)+\frac{1}{3^3}\sin^3\left(\frac{3^3\pi}{2}\right)+\ldots\ldots\ldots$

$$=\sum_{r=0}^{n}\frac{1}{3^r}\sin^3\left(3^r\frac{\pi}{2}\right)=\frac{1}{4}\left[3\sin\left(\frac{\pi}{2}\right)-\frac{1}{3^n}\sin\left(3^{n+1}\frac{\pi}{2}\right)\right]$$

As $n\to\infty$, $\frac{1}{3^n}\to 0\Rightarrow\sum_{r=0}^{n}\frac{1}{3^r}\sin^3\left(3^r\frac{\pi}{2}\right)\to\frac{3}{4}$ (shown)

5(i)



$$\overrightarrow{ON}=\frac{2\overrightarrow{OA}+\overrightarrow{OB}}{3}=\frac{2}{3}a+\frac{1}{3}b$$

$$\overrightarrow{OM}=\left(\overrightarrow{ON}\bullet\hat{b}\right)\hat{b}$$

$$=\frac{1}{|b|^2}\left[\left(\frac{2}{3}a+\frac{1}{3}b\right)\bullet b\right]b$$

$$=\frac{1}{3|b|^2}\left[(2a+b)\bullet b\right]b$$

$$=\frac{1}{3|b|^2}\left[2a\bullet b+|b|^2\right]b-----------(1)$$

Since $|a|=|b|$ and acute angle $AOB=\frac{\pi}{4}$,

$$a\bullet b=|a||b|\cos\frac{\pi}{4}=|a||b|\left(\frac{1}{\sqrt{2}}\right)=\frac{|b|^2}{\sqrt{2}}$$

Substituting this into (1) gives

$$\overrightarrow{OM}=\frac{1}{3|b|^2}\left[\frac{2|b|^2}{\sqrt{2}}+|b|^2\right]b=\frac{1}{3}\left(\sqrt{2}+1\right)b\quad\text{(shown)}$$

(ii) Area of triangle $OMN = \frac{1}{2}\left|\overrightarrow{OM} \times \overrightarrow{ON}\right|$

$$= \frac{1}{2}\left|\frac{1}{3}\left(\sqrt{2}+1\right)b \times \left(\frac{2}{3}a + \frac{1}{3}b\right)\right|$$

$$= \frac{1}{6}\left(\sqrt{2}+1\right)\left(\frac{1}{3}\right)\left|b \times (a+b)\right|$$

$$= \frac{\sqrt{2}+1}{18}\left|b \times a\right| = \frac{\sqrt{2}+1}{18}\,|\,b\,|\,|\,a\,|\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}+1}{18\sqrt{2}} = \frac{2+\sqrt{2}}{36} \text{ sq units (shown)} \qquad (\because\ |\,a\,| = |\,b\,| = 1\ )$$

6(a) $\int \frac{8x^2+1}{4x^2-1}dx = \int \frac{2\left(4x^2-1\right)+3}{4x^2-1}dx = \int 2 + \frac{3}{4x^2-1}dx$

$$= 2x + \frac{3}{2}\left(\frac{1}{2}\right)\ln\left|\frac{2x-1}{2x+1}\right| + C = 2x + \frac{3}{4}\ln\left|\frac{2x-1}{2x+1}\right| + C \text{ (shown)}$$

(b) $\int e^{3x}\tan^{-1}\left(e^{-3x}\right)dx = \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) - \int \frac{e^{3x}}{3} \bullet \frac{-3e^{-3x}}{1+\left(e^{-3x}\right)^2}dx$

$$= \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) + \int \frac{1}{1+e^{-6x}}dx$$

$$= \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) + \int \frac{e^{6x}}{e^{6x}\left(1+e^{-6x}\right)}dx$$

$$= \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) + \int \frac{e^{6x}}{e^{6x}+1}dx$$

$$= \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) + \frac{1}{6}\int \frac{6e^{6x}}{e^{6x}+1}dx$$

$$= \frac{e^{3x}}{3}\tan^{-1}\left(e^{-3x}\right) + \frac{1}{6}\ln\left(e^{6x}+1\right) + C \text{ (shown)}$$

7(i) $y=\left(4\cos^2\theta-1\right)\tan\theta=4\cos^2\theta\bullet\left(\frac{\sin\theta}{\cos\theta}\right)-\tan\theta$

$=4\sin\theta\cos\theta-\tan\theta=2\sin 2\theta-\tan\theta$

Differentiating the above $wrt\ \theta$ on both sides gives

$$\frac{dy}{d\theta}=4\cos 2\theta-\sec^2\theta$$

Differentiating $x=4\cos^2\theta-1$ $wrt\ \theta$ on both sides gives

$$\frac{dx}{d\theta}=-8\sin\theta\cos\theta$$

$$\frac{dy}{dx}=\frac{4\cos 2\theta-\sec^2\theta}{-8\sin\theta\cos\theta}=\frac{-4\cos 2\theta+\sec^2\theta}{8\sin\theta\cos\theta}$$

$$=\frac{-4(1-2\sin^2\theta)+\sec^2\theta}{8\sin\theta\cos\theta}=\frac{8\sin^2\theta+\sec^2\theta-4}{8\sin\theta\cos\theta}\ \text{(shown)}$$

At $\theta=0$, $\frac{dx}{dy}=\frac{8\sin\theta\cos\theta}{8\sin^2\theta+\sec^2\theta-4}=0\Rightarrow$ tangent to C is parallel to the $y-$axis.

(ii) $x=0\Rightarrow 4\cos^2\theta-1=0$

$\cos^2\theta=\frac{1}{4}$ or $\cos\theta=\pm\frac{1}{2}$

$\therefore\ \theta=-\frac{\pi}{3}$ or $\frac{\pi}{3}$

$y=0\Rightarrow\left(4\cos^2\theta-1\right)\tan\theta=0$

$\therefore\ \theta=-\frac{\pi}{3}\ 0$ or $\frac{\pi}{3}$

Hence, at the origin the values of θ are $-\frac{\pi}{3}$ or $\frac{\pi}{3}$ (shown)

(iii) When $x=3$, $\theta=0$.

Area= $2\int_0^3 y\,dx=2\int_{\frac{\pi}{3}}^{0} y\frac{dx}{d\theta}d\theta=2\int_{\frac{\pi}{3}}^{0}\left(4\cos^2\theta-1\right)\tan\theta\bullet\left(-8\sin\theta\cos\theta\right)d\theta$

$$= 16\int_0^{\frac{\pi}{3}} \left(4\cos^2\theta - 1\right)\tan\theta \bullet \left(\sin\theta\cos\theta\right)d\theta$$

$$= 16\int_0^{\frac{\pi}{3}} \left(4\sin\theta\cos\theta - \tan\theta\right)\bullet\left(\sin\theta\cos\theta\right)d\theta$$

$$= 16\int_0^{\frac{\pi}{3}} 4\sin^2\theta\cos^2\theta - \sin^2\theta \;\; d\theta$$

$$= 16\int_0^{\frac{\pi}{3}} \sin^2 2\theta - \sin^2\theta \; d\theta$$

$$= 16\int_0^{\frac{\pi}{3}} \frac{1-\cos 4\theta}{2} - \frac{1-\cos 2\theta}{2} \; d\theta$$

$$= \int_0^{\frac{\pi}{3}} 8\cos 2\theta - 8\cos 4\theta \; d\theta$$

$$= \left[4\sin 2\theta - 2\sin 4\theta\right]_0^{\frac{\pi}{3}} = 4\left(\frac{\sqrt{3}}{2}\right) - 2\left(-\frac{\sqrt{3}}{2}\right) = 3\sqrt{3} \text{ sq units (shown)}$$

(iv) Volume generated $= \pi\int_0^3 y^2 dx = \pi\int_{\frac{\pi}{3}}^0 y^2 \frac{dx}{d\theta} d\theta$

$$= \pi\int_{\frac{\pi}{3}}^0 \left(4\cos^2\theta - 1\right)^2 \tan^2\theta \bullet \left(-8\sin\theta\cos\theta\right) d\theta$$

$$= 8\pi\int_0^{\frac{\pi}{3}} \left(4\cos^2\theta - 1\right)^2 \tan^2\theta \bullet \left(\sin\theta\cos\theta\right) d\theta$$

= 16.0 cubic units (shown)

8. Let P_n be the proposition that $\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin\left[(2n+1)\theta\right] - \sin\theta}{2\sin\theta}$, $n \in Z^+$

For P_1 : $LHS = \cos 2\theta$; $RHS = \dfrac{\sin 3\theta - \sin\theta}{2\sin\theta} = \dfrac{2\cos 2\theta \sin\theta}{2\sin\theta} = \cos 2\theta$

Since $LHS = RHS$, P_1 is true.

Assume P_k is true for some $k \in \mathrm{Z}^+$, ie $\displaystyle\sum_{r=1}^{k} \cos(2r\theta) = \frac{\sin[(2k+1)\theta] - \sin\theta}{2\sin\theta}$

Considering P_{k+1} :

$$\sum_{r=1}^{k+1} \cos(2r\theta) = \sum_{r=1}^{k} \cos(2r\theta) + \cos[2(k+1)\theta]$$

$$= \frac{\sin[(2k+1)\theta] - \sin\theta}{2\sin\theta} + \cos[2(k+1)\theta]$$

$$= \frac{\sin[(2k+1)\theta] - \sin\theta + 2\cos[2(k+1)\theta]\sin\theta}{2\sin\theta}$$

$$= \frac{\sin[(2k+1)\theta] - \sin\theta + \sin[2(k+1)\theta + \theta] - \sin[2(k+1)\theta - \theta]}{2\sin\theta}$$

$$= \frac{\sin[(2k+1)\theta] - \sin\theta + \sin[(2k+3)\theta] - \sin[(2k+1)\theta]}{2\sin\theta}$$

$$= \frac{\sin[(2k+3)\theta] - \sin\theta}{2\sin\theta} = \frac{\sin[(2(k+1)+1)\theta] - \sin\theta}{2\sin\theta}$$

P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, by mathematical induction,

$$\sum_{r=1}^{n} \cos(2r\theta) = \frac{\sin[(2n+1)\theta] - \sin\theta}{2\sin\theta}, \quad n \in \mathrm{Z}^+ \quad \text{(shown)}$$

$$\sum_{r=1}^{n} \cos^2(r\theta) = \sum_{r=1}^{n} \frac{\cos(2r\theta)+1}{2} = \frac{1}{2}\sum_{r=1}^{n}[\cos(2r\theta)+1]$$

$$= \frac{1}{2} \bullet \frac{\sin[(2n+1)\theta] - \sin\theta}{2\sin\theta} + \frac{1}{2}n$$

$$= \frac{\sin[(2n+1)\theta] - \sin\theta}{4\sin\theta} + \frac{1}{2}n \quad \text{(shown)}$$

9. $\frac{d}{dx}\left[\cot\left(x^2\right)\right] = -2x\cos ec\left(x^2\right)$ (shown)

$$\int x^3 \cos ec\left(x^2\right) dx = \int x^2 \bullet x\cos ec\left(x^2\right)dx$$

$$= -\frac{1}{2}\int x^2 \bullet -2x\cos ec\left(x^2\right)dx$$

$$= -\frac{1}{2}\left\{\left[x^2 \cot\left(x^2\right)\right] - \int 2x\cot\left(x^2\right)dx\right\}$$

$$= -\frac{1}{2}x^2 \cot\left(x^2\right) + \int x\cot\left(x^2\right)dx$$

$$= -\frac{1}{2}x^2 \cot\left(x^2\right) + \frac{1}{2}\int \frac{2x\cos\left(x^2\right)}{\sin\left(x^2\right)}dx$$

$$= -\frac{1}{2}x^2 \cot\left(x^2\right) + \frac{1}{2}\ln\left|\sin\left(x^2\right)\right| + C \text{ (shown)}$$

10(i) $\int\left(\frac{e^x}{e^{2x}+1}\right)^2 dx = \int \frac{e^{2x}}{\left(e^{2x}+1\right)^2}dx$

$$= \int e^{2x}\left(e^{2x}+1\right)^{-2}dx = \frac{1}{2}\int 2e^{2x}\left(e^{2x}+1\right)^{-2}$$

$$= -\frac{1}{2}\left(e^{2x}+1\right)^{-1} + C = -\frac{1}{2}\left(\frac{1}{e^{2x}+1}\right) + C$$

By comparison, $A = -\frac{1}{2}$ (shown)

(ii) Volume generated $= \pi\int_0^{\ln 2} y^2 dx = \pi\int_0^{\ln 2} \frac{e^{2x}}{\left(e^{2x}+1\right)^2}dx$

$$= -\frac{1}{2}\pi\left[\left(\frac{1}{e^{2x}+1}\right)\right]_0^{\ln 2} = -\frac{1}{2}\pi\left(\frac{1}{5}-\frac{1}{2}\right) = \frac{3}{20}\pi \text{ cubic units (shown)}$$