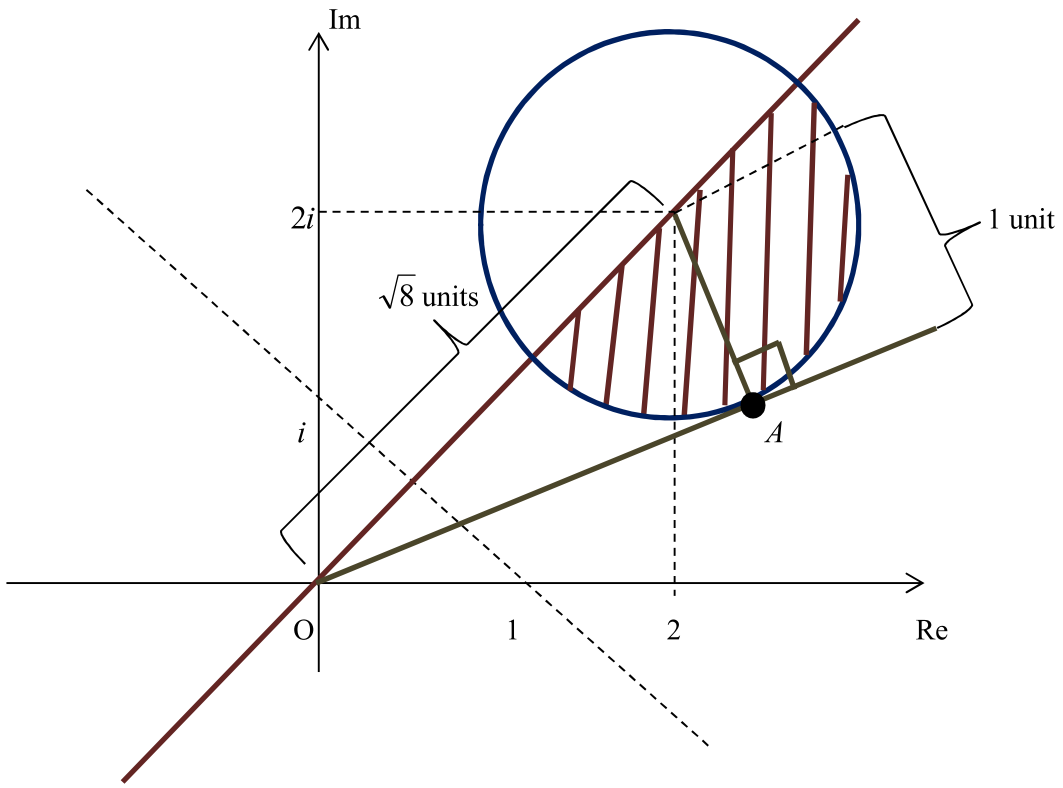
Additional Revision Questions 3 Solutions

1(i)



(ii) Based on the above Argand diagram, A denotes the point where $\arg(z)$ has the smallest value.

At this point, $|z| = OA = \sqrt{8-1} = \sqrt{7}$ units (shown)

2(a) Let $u = \tan x \Rightarrow x = \tan^{-1} u$

Then $\frac{dx}{du} = \frac{1}{1+u^2}$, ie $dx = \frac{1}{1+u^2} du$

When $x = \frac{\pi}{4}$, $u = 1$; when $x = 0$, $u = 0$

Hence, $\int_0^{\frac{\pi}{4}} (\tan^{n+2} x + \tan^n x) dx = \int_0^1 (u^{n+2} + u^n) \frac{1}{1+u^2} du$

$$\int_0^1 u^n (u^2+1) \frac{1}{1+u^2} du = \int_0^1 u^n \ du \ \left[\frac{u^{n+1}}{n+1}\right]_0^1 = \frac{1^{n+1}}{n+1} - 0 = \frac{1}{n+1} \quad \text{(shown)}$$

(b) $\frac{dy}{dx} = y(4-y) \Rightarrow \int \frac{1}{y(4-y)}\,dy = \int dx$

$$\int \frac{1}{4y} + \frac{1}{4(4-y)}\,dy = x + C$$

$$\frac{1}{4}\int \frac{1}{y} + \frac{1}{(4-y)}\,dy = x + C$$

$$\frac{1}{4}\left[\ln|y| - \ln|4-y|\right] = x + C$$

$$\ln|y| - \ln|4-y| = 4x + B \qquad [\because\ B = 4C]$$

$$\ln\left|\frac{y}{4-y}\right| = 4x + B$$

$$\frac{y}{4-y} = e^{4x+B} = e^{4x} \bullet e^{B} = Ae^{4x} \qquad [\because\ A = e^{B}]$$

$$\frac{4-y}{y} = De^{-4x} \qquad [\because\ D = \frac{1}{A}]$$

$$\frac{4}{y} - 1 = De^{-4x}$$

$$\frac{4}{y} = De^{-4x} + 1$$

$\therefore\quad y = \frac{4}{De^{-4x} + 1}$ (shown)

3(i) $\frac{dN}{dt} = \frac{N(1800-N)}{3600} \rightarrow \int \frac{3600}{N(1800-N)} = \int dt$

$$3600\int \frac{1}{1800N} + \frac{1}{1800(1800-N)} = \int dt$$

$$2\int \frac{1}{N} + \frac{1}{(1800-N)} = \int dt$$

$$2\left[\ln|N| - \ln|1800-N|\right] = t + C$$

$$2\ln\left|\frac{N}{1800-N}\right| = t + C ----------(1)$$

When $N = 300,\ t = 0 \Rightarrow\ C = 2\ln\left|\frac{300}{1500}\right| = 2\ln\left(\frac{1}{5}\right)$

Therefore, (1) becomes $2\ln\left|\frac{N}{1800-N}\right| = t + 2\ln\left(\frac{1}{5}\right)$

$$2\left[\ln\left|\frac{N}{1800-N}\right| - \ln\left(\frac{1}{5}\right)\right] = t$$

$$2\ln\left|\frac{5N}{1800-N}\right| = t$$

$$\ln\left|\frac{5N}{1800-N}\right| = \frac{t}{2}$$

$\frac{5N}{1800-N} = e^{\frac{t}{2}} \Rightarrow \frac{1800-N}{5N} = e^{-\frac{t}{2}}$

$$\frac{360}{N} - \frac{1}{5} = e^{-\frac{t}{2}}$$

$$\therefore\quad N = \frac{360}{\frac{1}{5} + e^{-\frac{t}{2}}} = \frac{1800}{1 + 5e^{-\frac{t}{2}}} \quad \text{(shown)}$$

(ii) When $t \to \infty,\ e^{-\frac{t}{2}} \to 0$ and $N \to \frac{1800}{1} = 1800$ birds (shown)

4(i) Angle $BAC = \pi - \frac{3\pi}{4} - \theta = \frac{\pi}{4} - \theta$

$$\frac{1}{\sin\left(\frac{\pi}{4} - \theta\right)} = \frac{AC}{\sin\frac{\pi}{4}} \Rightarrow AC = \frac{\sin\frac{\pi}{4}}{\sin\left(\frac{\pi}{4} - \theta\right)}$$

$$AC = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta} = \frac{1}{\cos\theta - \sin\theta} \quad \text{(shown)}$$

(ii) $\dfrac{1}{\cos\theta-\sin\theta}\approx\dfrac{1}{1-\dfrac{\theta^2}{2}-\theta}=\left[1-\left(\theta+\dfrac{\theta^2}{2}\right)\right]^{-1}$

$$=1+\left(\theta+\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(\theta+\frac{\theta^2}{2}\right)^2+\text{.........}$$

$$\approx 1+\left(\theta+\frac{\theta^2}{2}\right)+\theta^2=1+\theta+\frac{3\theta^2}{2}\text{ , where } a=1,\ b=\frac{3}{2}\ \text{(shown)}$$

5(i) Differentiating $x-y=(x+y)^2$ wrt x on both sides,

$$1-\frac{dy}{dx}=2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$1-\frac{dy}{dx}=2\left(x+y+\frac{1}{2}\right)\left(1+\frac{dy}{dx}\right)-2\left(\frac{1}{2}\right)\left(1+\frac{dy}{dx}\right)$$

$$1-\frac{dy}{dx}+\left(1+\frac{dy}{dx}\right)=2\left(x+y+\frac{1}{2}\right)\left(1+\frac{dy}{dx}\right)$$

$$2=(2x+2y+1)\left(1+\frac{dy}{dx}\right)$$

$$\therefore\ 1+\frac{dy}{dx}=\frac{2}{2x+2y+1}\ \text{(shown)}$$

(ii) Differentiating $1+\dfrac{dy}{dx}=\dfrac{2}{2x+2y+1}$. wrt x on both sides,

$$\frac{d^2y}{dx^2}=-2(2x+2y+1)^{-2}\left(2+2\frac{dy}{dx}\right)=-\frac{2}{(2x+2y+1)^2}\left(1+\frac{dy}{dx}\right)(2)$$

$$=-\frac{4}{(2x+2y+1)^2}\left(1+\frac{dy}{dx}\right)=-\left(\frac{2}{2x+2y+1}\right)^2\left(1+\frac{dy}{dx}\right)$$

$$=-\left(1+\frac{dy}{dx}\right)^3\ \text{(shown)}$$

(iii) When the curve has a turning point, $\dfrac{dy}{dx}=0\Rightarrow\dfrac{d^2y}{dx^2}=-(1+0)^3=-1<0$

Therefore, the turning point is a **maximum** point. (shown)

6(i) $z^3 = (1+id)^3 = 1+3(id)+3(id)^2+(id)^3$

$= 1+3id-3d^2-id^3 = \left(1-3d^2\right)+\left(3d-d^3\right)i$ (shown)

(ii) If z^3 is real, then $3d-d^3=0 \Rightarrow d\left(3-d^2\right)=0$

Hence, $d=\pm\sqrt{3}$ and $z=1+\sqrt{3}i$ or $1-\sqrt{3}i$ (shown)

(iii) Choosing $z=1-\sqrt{3}i$, we have $z^n=\left(2e^{-i\frac{\pi}{3}}\right)^n=2^n e^{-i\frac{n\pi}{3}}$

$|z^n|>1000 \Rightarrow \left|2^n e^{-i\frac{n\pi}{3}}\right|>1000,\quad \text{ie}\quad 2^n>1000$

$\therefore\ n>\frac{\ln 1000}{\ln 2}=9.966 \Rightarrow n_{\min}=10$ (shown)

Modulus of $z^n=2^{10}=1024$, argument of $z^n=-\frac{10\pi}{3}=\frac{2\pi}{3}$ (shown)

7. Let the amounts invested in A, B and C be x, y and z respectively.

Then $x+y+z=50000$ ---------(1)

$0.06x+0.08y+0.1z=3700$ ----------(2)

$x=2z \Rightarrow x-2z=0$ ----------(3)

Based on the above 3 equations, the augmented matrix formed is given by

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 50000 \\ 0.06 & 0.08 & 0.1 & 3700 \\ 1 & 0 & -2 & 0 \end{array}\right)$$

Solving this matrix gives $x=\$30,000$, $y=\$5000$, $z=\$15000$ (shown)

8(i) $u_2=\frac{3(2)-1}{6}=\frac{5}{6}$, $u_2=\frac{3\left(\frac{5}{6}\right)-1}{6}=\frac{1}{4}$ (shown)

(ii) As $n\to\infty$, $l=\frac{3l-1}{6} \Rightarrow 6l=3l-1$

$\therefore\ l = -\frac{1}{3}$ (shown)

(iii) Let P_n be the hypothesis that $u_n = \frac{14}{3}\left(\frac{1}{2}\right)^n - \frac{1}{3}$, where $n \in \mathbb{Z}^+$.

For P_1, $LHS = u_1 = 2$, $RHS = u_1 = \frac{14}{3}\left(\frac{1}{2}\right) - \frac{1}{3} = \frac{7}{3} - \frac{1}{3} = 2$

$LHS = RHS$, $\therefore$ P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, ie $u_k = \frac{14}{3}\left(\frac{1}{2}\right)^k - \frac{1}{3}$.

Then $u_{k+1} = \frac{3u_k - 1}{6} = \frac{1}{2}u_k - \frac{1}{6}$

$$= \frac{1}{2}\left[\frac{14}{3}\left(\frac{1}{2}\right)^k - \frac{1}{3}\right] - \frac{1}{6}$$

$$= \frac{14}{3}\left(\frac{1}{2}\right)^{k+1} - \frac{1}{6} - \frac{1}{6} = \frac{14}{3}\left(\frac{1}{2}\right)^{k+1} - \frac{1}{3}$$

Therefore, P_k is true $\Rightarrow$ P_{k+1} is true.

Since P_1 is true, by mathematical induction, $u_n = \frac{14}{3}\left(\frac{1}{2}\right)^n - \frac{1}{3}$, for $n \in \mathbb{Z}^+$. (shown)

9. For $u = 2x + y$, differentiating both sides *wrt* x gives

$$\frac{du}{dx} = 2 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$$

$\frac{dy}{dx} = \sin^2(2x + y) - 3$ becomes

$$\frac{du}{dx} - 2 = \sin^2 u - 3$$

$\frac{du}{dx} = \sin^2 u - 1 = -\cos^2 u$ (shown)

$$-\int \frac{1}{\cos^2 u}\, du = \int dx$$

$$-\int \sec^2 u \;\; du = \int dx$$

$$-\tan u = x + C$$

$$-\tan(2x + y) = x + C$$

Since the particular solution curves passes through the origin, ie when $\underline{x = 0,\;\; y = 0,}$ substituting these values into the immediate above equation gives $C = 0.$

$$\therefore \quad -\tan(2x + y) = x$$

$$2x + y = \tan^{-1}(-x) \Rightarrow y = \tan^{-1}(-x) - 2x \text{ (shown)}$$

10. Differentiating $y = \ln(\cos 2x)$ wrt x on both sides,

$$\frac{dy}{dx} = \frac{-2\sin 2x}{\cos 2x} = -2\tan 2x$$

Differentiating this once again wrt x on both sides,

$$\frac{d^2 y}{dx^2} = -4\sec^2 2x = -4\left(1 + \tan^2 2x\right)$$

$$= -4\left(1 + \left[-\frac{1}{2}\left(\frac{dy}{dx}\right)\right]^2\right) = -4\left[1 + \frac{1}{4}\left(\frac{dy}{dx}\right)^2\right] = -4 - \left(\frac{dy}{dx}\right)^2 \text{ (shown)}$$

Differentiating this result once again wrt x on both sides,

$$\frac{d^3 y}{dx^3} = -2\left(\frac{dy}{dx}\right)\left(\frac{d^2 y}{dx^2}\right).$$

Doing this for a final round,

$$\frac{d^4 y}{dx^4} = -2\left[\left(\frac{d^2 y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)\left(\frac{d^3 y}{dx^3}\right)\right]$$

When $x = 0,\;\; y = 0,\;\; \frac{dy}{dx} = 0,\;\; \frac{d^2 y}{dx^2} = -4,\;\; \frac{d^3 y}{dx^3} = 0$ and $\frac{d^4 y}{dx^4} = -32$

Hence, $y \approx -4\left(\frac{x^2}{2!}\right) - 32\left(\frac{x^4}{4!}\right) = -2x^2 - \frac{4x^4}{3}$ (shown)

$$\ln(\sec^2 2x) = \ln\left(\frac{1}{\cos^2 2x}\right) = -\ln(\cos 2x)^2 = -2\ln(\cos 2x)$$

$$\approx -2\left(-2x^2 - \frac{4x^4}{3}\right) = 4x^2 + \frac{8x^4}{3} \text{ (shown)}$$