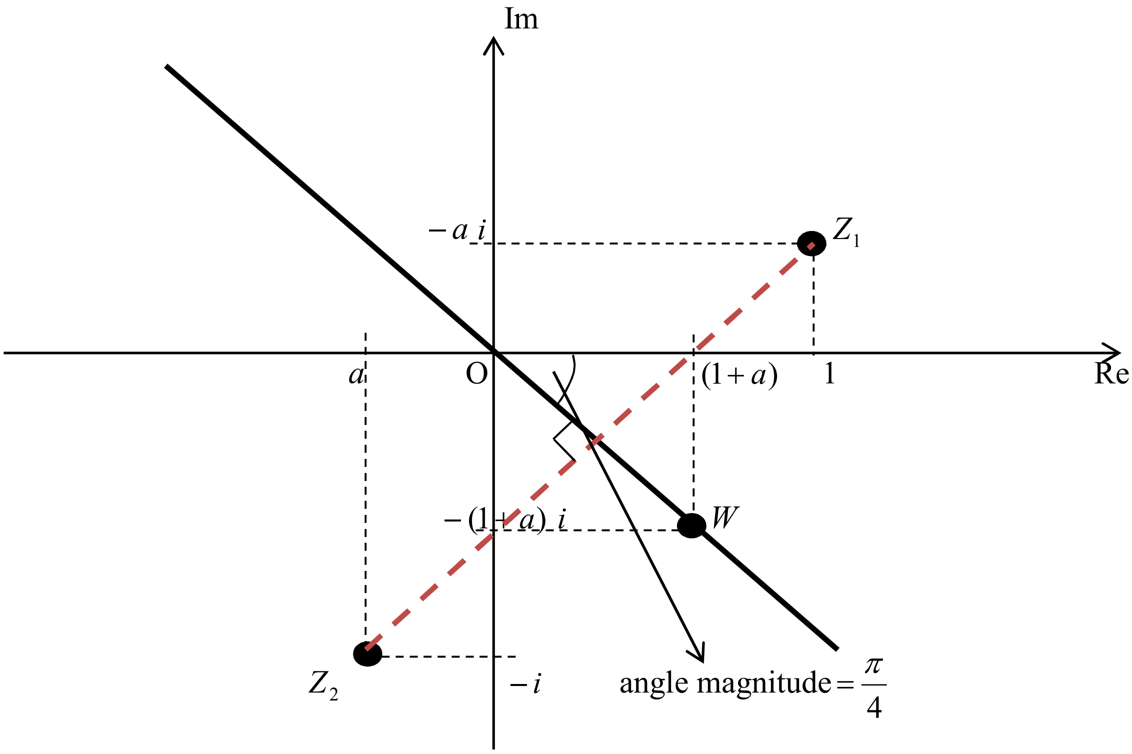
Additional Complex Number Problems Solutions

1. $w = z_1 + z_2 = 1 - ai + a - i = (1+a) - (1+a)\ i$, whereby $p = 1 + a$ and $q = -(1+a)$ (shown)

(i) $|w| = \sqrt{(1+a)^2 + [-(1+a)]^2} = \sqrt{2}(1+a)$ (shown)

(ii) $\arg(w) = -\frac{\pi}{4}$ (shown)

(iii)



The point W lies on the **perpendicular bisector** of the line joining Z_1 and Z_2. (shown)

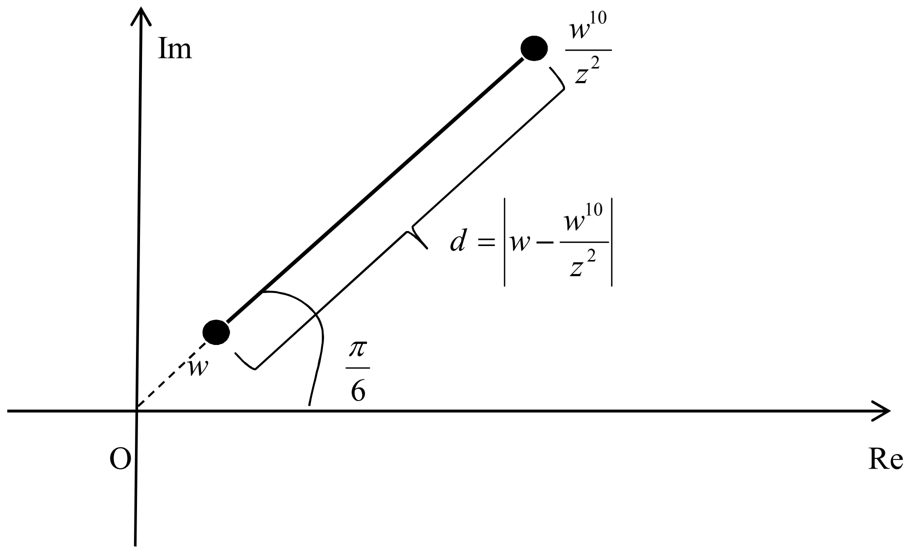
(iv) $\arg\left(\frac{1}{w^n}\right) = -n\ \arg(w) = \frac{n\pi}{4}$

$\therefore$ Smallest integer value of $n = 3$

such that $\arg\left(\frac{1}{w^n}\right) = \frac{3\pi}{4}$, ie point P lies in 2nd quadrant on the bolded black line. (shown)

2 (a) $\left|\frac{w^{10}}{z^2}\right| = \frac{|w|^{10}}{|z|^2} = \frac{2^{10}}{(\sqrt{2})^2} = 2^9 = 512$

$\arg\left(\frac{w^{10}}{z^2}\right) = 10\ \arg(w) - 2\arg(z) = 10\left(\frac{\pi}{6}\right) - 2\left(\frac{3\pi}{4}\right) = \frac{\pi}{6}$ (shown)



From the above Argand diagram, it can be observed that

$$d = \left| w - \frac{w^{10}}{z^2} \right| = 512 - 2 = 510 \quad \text{(shown)}$$

(b) $\frac{1}{e^{iz}} = 2 + i \Rightarrow \frac{1}{e^{i(a+ib)}} = 2 + i \Rightarrow \frac{1}{e^{ia-b}} = 2 + i$

$$e^{b} e^{-ia} = 2 + i$$

$$e^{b}(\cos a - i \sin a) = 2 + i$$

$$e^{b}\left[\cos(-a) + i \sin(-a)\right] = 2 + i$$

Hence, $\quad -a = \arg(2+i) = \tan^{-1}\left(\frac{1}{2}\right) \to a = -\tan^{-1}\left(\frac{1}{2}\right)$

and $\quad e^{b} = |2 + i| = \sqrt{5} \to b = \ln\sqrt{5} = \frac{1}{2}\ln 5$ (shown)

3(a) $z = e^{\frac{\alpha}{2i}} + i = e^{-\frac{\alpha}{2}i} + i = \cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right) + i = \cos\left(\frac{\alpha}{2}\right) + (i)\left[1 - \sin\left(\frac{\alpha}{2}\right)\right]$

$|z| = \sqrt{3} \Rightarrow \sqrt{\cos^{2}\left(\frac{\alpha}{2}\right) + \left[1 - \sin\left(\frac{\alpha}{2}\right)\right]^{2}} = \sqrt{3}$

$$\cos^{2}\left(\frac{\alpha}{2}\right) + \left[1 - \sin\left(\frac{\alpha}{2}\right)\right]^{2} = 3$$

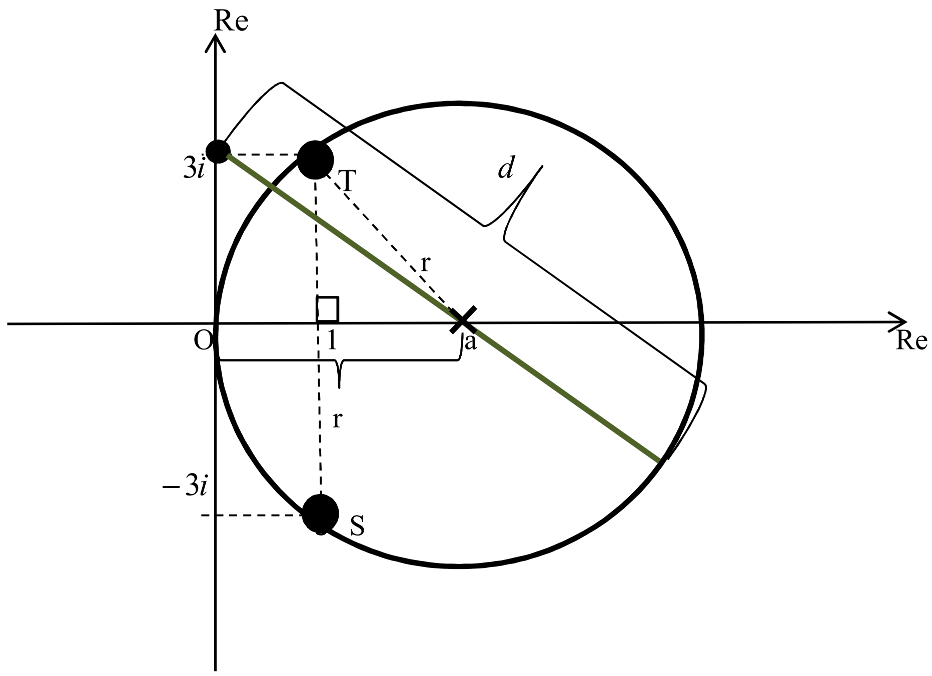
$$\cos^2\left(\frac{\alpha}{2}\right)+1-2\sin\left(\frac{\alpha}{2}\right)+\sin^2\left(\frac{\alpha}{2}\right)=3$$

$$2-2\sin\left(\frac{\alpha}{2}\right)=3\rightarrow\sin\left(\frac{\alpha}{2}\right)=-\frac{1}{2}$$

$$\therefore\ \frac{\alpha}{2}=-\frac{\pi}{6}\Rightarrow\alpha=-\frac{\pi}{3}\quad\text{(shown)}$$

(b)(i) $|z_1|=\sqrt{10},\ \ \arg(z_1)=-\tan^{-1}3=-1.249$ (shown)

(ii)



From the above Argand diagram, we have

$$r^2=(r-1)^2+3^2\Rightarrow r=5$$

Also, $a=r=5$ (as evidenced from the fact that the distance from centre of the circle to the origin is also the radius of the circle)

Hence, equation of circle is $|z-5|=5$ (shown)

(c) Maximum value of $|z-3i|=d=5+\sqrt{3^2+5^2}=5+\sqrt{34}$ (shown)

4(i) $z=r(\cos\theta+i\sin\theta)=re^{i\theta}\rightarrow z^2=r^2e^{i2\theta},\ z^*=re^{-i\theta}$

$$\therefore\ \frac{z^2}{z^*} = \frac{r^2 e^{i2\theta}}{re^{-i\theta}} = re^{i3\theta} = r(\cos 3\theta + i\sin 3\theta) \quad \text{(shown)}$$

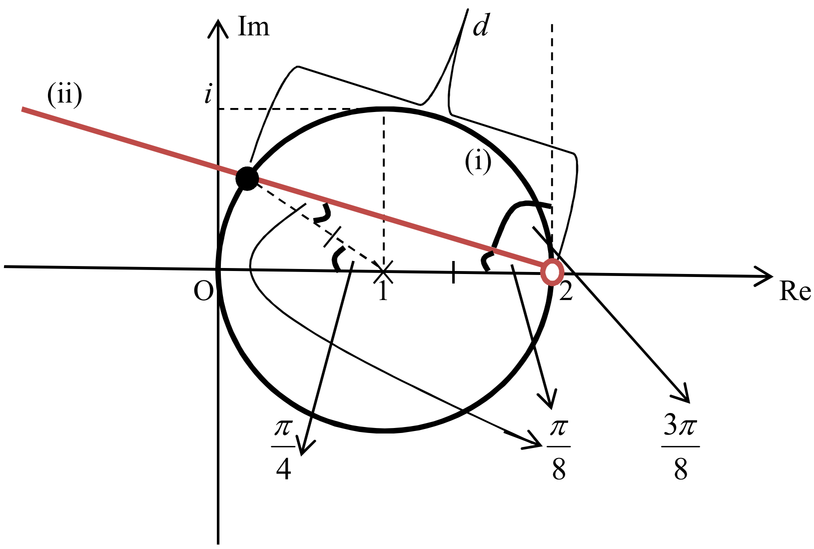
(ii) $z^2 = iz^* \Rightarrow \dfrac{z^2}{z^*} = i$

$$r(\cos 3\theta + i\sin 3\theta) = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right],\ k = 0, \pm 1$$

By comparison, $r = 1$, $3\theta = -\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2} \Rightarrow \theta = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ (shown)

(Note that other integer values of k selected would cause θ to fall outside the standard range.)

5.



From the above Argand diagram, point of intersection is given by

$$z = \left(1 - \cos\frac{\pi}{4}\right) + i\sin\frac{\pi}{4} = \left(1 - \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}} \quad \text{(shown)}$$

Recognising that this point of intersection can also be written as

$z = (2 - d\sin\dfrac{3\pi}{8}) + i\left(d\cos\dfrac{3\pi}{8}\right)$, by comparison with the previous expression we have

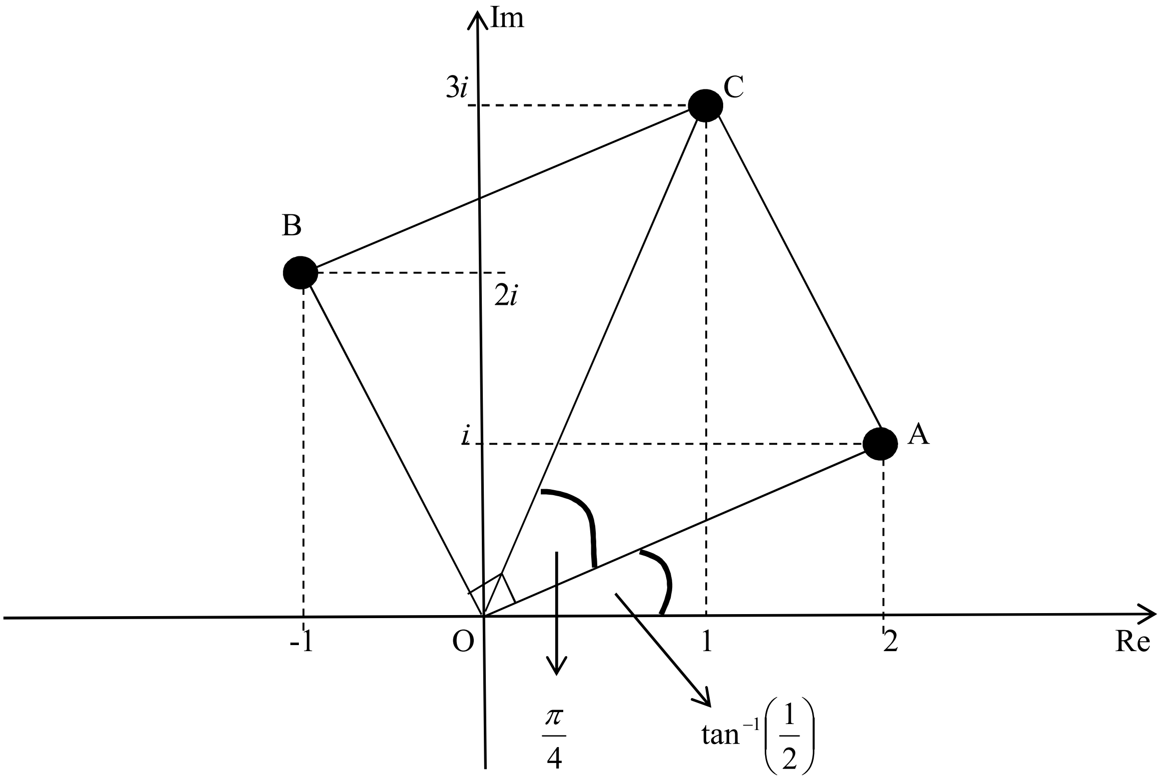
$$2 - d\sin\frac{3\pi}{8} = 1 - \frac{1}{\sqrt{2}} \Rightarrow d\sin\frac{3\pi}{8} = 1 + \frac{1}{\sqrt{2}} \text{-------}(1)$$

$$d\cos\frac{3\pi}{8} = \frac{1}{\sqrt{2}} \text{-------}(2)$$

$(1)\div(2):\ \tan\frac{3\pi}{8}=\left(1+\frac{1}{\sqrt{2}}\right)\div\frac{1}{\sqrt{2}}=\sqrt{2}+1$ (shown)

6(i) $|z_1|=|z_2|=\sqrt{5},\quad \arg(z_1)=\tan^{-1}\left(\frac{1}{2}\right),\quad \arg(z_2)=\pi-\tan^{-1}(2)$ (shown)

(ii) Since $(i)(2+i)=-1+2i$, OA is perpendicular to OB. (shown)



Since OABC is a square, angle BOA=$\frac{\pi}{4}$

Hence, angle between OC and real axis is $=\frac{\pi}{4}+\tan^{-1}\left(\frac{1}{2}\right)$ (shown)

$z_1+z_2=1+3i,\qquad \arg(z_1+z_2)=\tan^{-1}3$

$\therefore\ \tan^{-1}3=\frac{\pi}{4}+\tan^{-1}\left(\frac{1}{2}\right)$ (shown)