Additional Calculus Problems (Integration And Applications)

Solutions

1(a) $\int \frac{1+x}{1-x}\,dx = \int \frac{-(1-x)+2}{1-x}\,dx = \int -1+\frac{2}{1-x}\,dx = -x - 2\ln|1-x| + C$ (shown)

1(b) $\int \frac{1}{\sqrt{x}-\sqrt{x-1}}\,dx = \int \frac{\sqrt{x}+\sqrt{x-1}}{(-1)}\,dx = \int -\sqrt{x}+\sqrt{x+1}\,dx = -\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$ (shown)

1(c) $\int \left(e^t + \frac{2}{e^t}\right)^2 dt = \int e^{2t} + 4 + \frac{4}{e^{2t}}\,dt = \frac{1}{2}e^{2t} + 4t - 2e^{-2t} + C$ (shown)

1(d) $\int \frac{1}{e^{2t}-1}\,dt = \int \frac{e^{2t}(e^{-2t})}{e^{2t}\left(1-e^{-2t}\right)}dt = \frac{1}{2}\int \frac{2e^{-2t}}{1-e^{-2t}}\,dt = \frac{1}{2}\ln\left|1-e^{-2t}\right| + C$ (shown)

1(e) $\int \sin\theta(1+\cos\theta)d\theta = -\int -\sin\theta(1+\cos\theta)d\theta = -\frac{1}{2}(1+\cos\theta)^2 + C$ (shown)

1(f) $\int \frac{12}{4t^2+8t-5}dt = \int \frac{3}{t^2+2t-\frac{5}{4}}\,dt = \int \frac{3}{(t+1)^2-\left(\frac{3}{2}\right)^2}dt = \frac{1}{2\left(\frac{3}{2}\right)}(3)\ln\left|\frac{t+1-\frac{3}{2}}{t+1+\frac{3}{2}}\right| + C$

$= \ln\left|\frac{t-\frac{1}{2}}{t+\frac{5}{2}}\right| + C = \ln\left|\frac{2t-1}{2t+5}\right| + C$ (shown)

1(g) $\int \frac{2+t^2}{(1-t)^2}dt = \int \frac{2+t^2}{t^2-2t+1}dt = \int \frac{\left(t^2-2t+1\right)-(2t+2)+3}{t^2-2t+1}dt = \int 1-\frac{2t+2}{t^2-2t+1}+\frac{3}{(1-t)^2}dt$

$= t - 2\ln|1-t| - \frac{3}{1-t} + C$ (shown)

1(h) $\int \theta(\cos\theta+\sin\theta)^2\,d\theta = \int \theta\left(\cos^2\theta + 2ins\theta\cos\theta + \sin^2\theta\right)d\theta = \int \theta(1+\sin 2\theta)d\theta$

$= \frac{\theta^2}{2} + \int \theta(\sin 2\theta)d\theta = \frac{\theta^2}{2} + \left(-\frac{1}{2}\cos 2\theta\right)(\theta) + \frac{1}{2}\int \cos 2\theta\,d\theta$

$= \frac{\theta^2}{2} - \frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta + C$ (shown)

1(i) $\int \ln\left(\frac{1}{x+1}\right)dx = \int \ln 1 - \ln(x+1)\,dx = -\int \ln(x+1)\,dx = -\left[x\ln(x+1) - \int x\left(\frac{1}{x+1}\right)dx\right]$

$$= -\left[x\ln(x+1) - \int \frac{x+1-1}{x+1}dx\right] = -\left[x\ln(x+1) - \int dx + \int \frac{1}{x+1}dx\right]$$

$$= -x\ln(x+1) + x - \ln|x+1| + C \text{ (shown)}$$

2. $$\int_0^4 \frac{2}{\sqrt{x}+1}dx = \int_1^5 \frac{2}{u}\left(\frac{1}{2}\right)(u-1)du = \int_1^5 1 - \frac{1}{u}du = [u - \ln|u|]_1^5 = 4 - \ln 5 \text{ (shown)}$$

$$\left[\because u = 2\sqrt{x}+1 \Rightarrow x = \frac{(u-1)^2}{4}, \frac{dx}{du} = \frac{1}{2}(u-1)\right]$$

3(a) $$A = \int_0^1 \frac{4x}{1+x^2}dx - \frac{1}{2}(1)(2) = 2\left[\ln|1+x^2|\right]_0^1 - 1 = 0.386 \text{ sq units (shown)}$$

3(b) $$V = \pi\int_0^1 \left(\frac{4x}{1+x^2}\right)^2 dx - \frac{1}{3}\pi(2)^2(1) = \pi\int_0^1 \frac{16x^2}{(1+x^2)^2}dx - \frac{4}{3}\pi = \pi\int_0^{\frac{\pi}{4}} \frac{16\tan^2\theta}{\sec^4\theta}(\sec^2\theta)d\theta - \frac{4}{3}\pi$$

$$= \pi\int_0^{\frac{\pi}{4}} \frac{16\tan^2\theta}{\sec^2\theta}d\theta - \frac{4}{3}\pi = 16\pi\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta - \frac{4}{3}\pi$$

$$= 16\pi\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2} d\theta - \frac{4}{3}\pi = 8\pi\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} - \frac{4}{3}\pi = 2.98 \text{ cubic units (shown)}$$

4 (i) $$\int_0^1 \sqrt{\frac{1}{3}(4-x^2)}dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{1}{3}}\sqrt{4-4\sin^2\theta}(2\cos\theta)d\theta = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{6}} (2\cos\theta)(2\cos\theta)d\theta$$

$$= \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{6}} 4\cos^2\theta \, d\theta = \frac{2}{\sqrt{3}}\int_0^{\frac{\pi}{6}} 1 + \cos 2\theta \, d\theta$$

$$= \frac{2}{\sqrt{3}}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3\sqrt{3}} + \frac{1}{2} \text{ (shown)}$$

4(ii) (a) $$A = \int_0^1 \sqrt{\frac{1}{3}(4-x^2)}dx + \int_1^{1.3} \frac{1}{x}dx = \frac{\pi}{3\sqrt{3}} + \frac{1}{2} + [\ln|x|]_1^{1.3} = 1.37 \text{ sq units (shown)}$$

(b) $$V = \pi\int_0^1 \frac{1}{3}(4-x^2)dx + \pi\int_1^{1.3} \frac{1}{x^2}dx = \frac{\pi}{3}\left[4x - \frac{x^3}{3}\right]_0^1 + \pi\left[\left(-\frac{1}{x}\right)\right]_1^{1.3} = 4.56 \text{ cubic units}$$

(shown)

5(a) Let $f(x) = xe^{|x|}$; since $f(-x) = -xe^{|-x|} = -xe^{|x|} = -f(x)$, $\int_{-2}^{2} xe^{|x|} = 0$ (shown)

(Note that the above shorter method is only feasible if the integration limits are **symmetrical**)

5(b) $\int \frac{-8x-3}{\sqrt{1-(2x+1)^2}}\,dx = \int \frac{-8x-3}{\sqrt{-4x^2-4x}}\,dx = \int \frac{(-8x-4)+1}{\sqrt{-4x^2-4x}}\,dx$

$$= \int \frac{-8x-4}{\sqrt{-4x^2-4x}}\,dx + \int \frac{1}{\sqrt{1-(2x+1)^2}}\,dx$$

$$= 2\sqrt{-4x^2-4x} + \frac{1}{2}\sin^{-1}(2x+1) + C \text{ (shown)}$$

6(i) $\int x^n \ln x \; dx = \left(\frac{x^{n+1}}{n+1}\right)(\ln x) - \int \left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{x}\right)dx = \left(\frac{x^{n+1}}{n+1}\right)(\ln x) - \int \frac{x^n}{n+1}\,dx$

$$= \left(\frac{x^{n+1}}{n+1}\right)(\ln x) - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C \text{ (shown)}$$

6(ii) $\frac{d}{dx}\left[\int x^n \ln x dx\right] = \frac{d}{dx}\left\{\frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C\right\}$

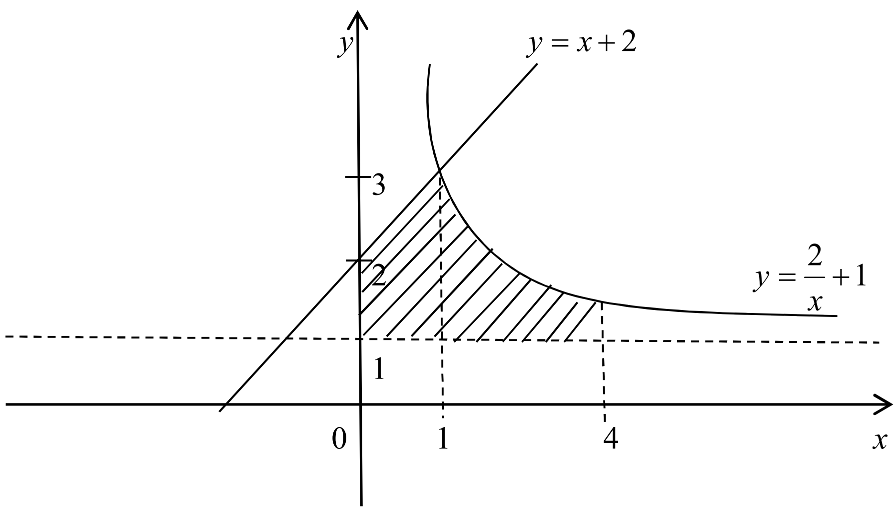
Let $n = 99$, then $x^{99} \ln x = \frac{x^{100}}{100^2}\left(\ln x^{100} - 1\right)$

$$x^{100}\left(\ln x^{100} - 1\right) = 100^2 x^{99} \ln x = x^{99}\left(\ln x^{10000}\right)$$

where $a = 99$ and $b = 10000$ (shown)

6(iii) $\int x^n \ln x^n dx = n\int x^n \ln x \; dx = \frac{nx^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C$ (shown)

7.



$$\text{Area} = \frac{1}{2}(1+2)(1) + \int_1^4 \frac{2}{x} + 1 \; dx - 3(1) = \frac{3}{2} + \left[2\ln|x| + x\right]_1^4 - 3 = 2\ln 4 + \frac{3}{2} \text{ sq units (shown)}$$

8(i) $16x^2 + y^2 = 3 \Rightarrow y = \sqrt{3-16x^2}$

$$\therefore \; A = \int_0^{\frac{\sqrt{3}}{4}} \sqrt{3-16x^2} \; dx \text{ (shown)}$$

8(ii) $$A = \int_0^{\frac{\pi}{2}} \sqrt{3 - 16\left(\frac{3}{16}\sin^2\theta\right)}\left(\frac{\sqrt{3}}{4}\cos\theta\right)d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{4}\cos^2\theta \; d\theta$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta = \frac{3}{4}\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{16} \text{ sq units (shown)}$$

8(iii) $$V = \pi \int_0^{\sqrt{3}} x^2 dy = \pi \int_0^{\sqrt{3}} \frac{3-y^2}{16} dy = \frac{\pi}{16}\left[3y - \frac{y^3}{3}\right]_0^{\sqrt{3}} = \frac{\pi}{16}\left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right)$$

$$= \frac{\pi}{16}(2\sqrt{3}) = \frac{\sqrt{3}\pi}{8} \text{ cubic units (shown)}$$