Additional Calculus Problems 3-Integration and Differential Equations

Solutions

1(a) $\int \frac{x^4 - x^2 + 2}{x^2(x-1)} dx = \int \frac{x^4 - x^2 + 2}{x^3 - x^2} dx = \int \frac{x(x^3 - x^2) + (x^3 - x^2) + 2}{x^3 - x^2} dx$

$$= \int x + 1 + \frac{2}{x^3 - x^2} \, dx \quad = \int x + 1 + \frac{2}{x^2(x-1)} \, dx$$

$$= \frac{x^2}{2} + x + 2\int \frac{1}{x-1} - \frac{1}{x^2} - \frac{1}{x} dx$$

$$= \frac{x^2}{2} + x + 2\left(\ln|x-1| + \frac{1}{x} - \ln|x| \right) + C$$

$$= \frac{x^2}{2} + x + \frac{2}{x} - 2\ln|x| + 2\ln|x-1| + C \quad \text{(shown)}$$

1(b) $\int \frac{\cos x}{\sin^2 x - \sin x - 2} dx = \int \frac{\cos x}{\left(\sin x - \frac{1}{2}\right)^2 - \frac{9}{4}} dx = \int \frac{\cos x}{\left(\sin x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$

$$= \frac{1}{2\left(\frac{3}{2}\right)} \ln\left| \frac{\sin x - \frac{1}{2} - \frac{3}{2}}{\sin x - \frac{1}{2} + \frac{3}{2}} \right| + C = \frac{1}{3} \ln\left| \frac{\sin x - 2}{\sin x + 1} \right| + C \quad \text{(shown)}$$

[Note that $\int \frac{f'(x)}{[f(x)]^2 - a^2} dx = \frac{1}{2a} \ln\left| \frac{f(x) - a}{f(x) + a} \right| + C$, where $f(x) = \sin x - \frac{1}{2}$]

2(a) $y^3 \frac{dy}{dx} + x^3 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}$

$$\int y^3 dy = -\int x^3 dx$$

$\frac{y^4}{4} = -\frac{x^4}{4} + B \rightarrow x^4 + y^4 = 4B = C$ (shown)

$$\frac{y^4}{4} = -\frac{x^4}{4} + B \rightarrow x^4 + y^4 = 4B = C \text{ (shown)}$$

*2(b) $\frac{dy}{dx} = \frac{x^2 + y^2}{xy} \Rightarrow \frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$ ---------(1)

Let $u = \frac{x}{y}$, then $\frac{du}{dx} = \frac{y - x\frac{dy}{dx}}{y^2} \Rightarrow y^2\frac{du}{dx} = y - x\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y^2}{x}\right)\frac{du}{dx} = \frac{1}{u} - \left(\frac{y}{u}\right)\frac{du}{dx} = \frac{1}{u} - \left(\frac{x}{u^2}\right)\frac{du}{dx}$$

[$\because \frac{y^2}{x} = y\left(\frac{y}{x}\right) = y\left(\frac{1}{u}\right) = \left(\frac{x}{u}\right)\left(\frac{1}{u}\right) = \frac{x}{u^2}$]

Substituting all these into (1) gives $\frac{1}{u} - \left(\frac{x}{u^2}\right)\frac{du}{dx} = u + \frac{1}{u}$

Multiplying both sides by u^2 gives

$$u - x\frac{du}{dx} = u^3 + u$$

$$-x\frac{du}{dx} = u^3$$

$-\int \frac{1}{x}\,dx = \int \frac{1}{u^3}\,du$ (This is the variables separable part)

$$-\ln|x| = -\frac{1}{2u^2} + B \rightarrow 2\ln|x| = \frac{1}{u^2} - 2B$$

$$2\ln|x| = \frac{y^2}{x^2} - 2B$$

$$y^2 = x^2(2\ln|x| + 2B) = x^2(2\ln|x| + C) = x^2(\ln x^2 + C)$$

$\therefore\ y = \pm x\sqrt{(\ln x^2 + C)}$ (shown)

2(c) $\dfrac{dy}{dx} = (y+4x)^2$ ------(1)

Let $u = y + 4x$, then $\dfrac{du}{dx} = \dfrac{dy}{dx} + 4 \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 4$

(1) becomes $\dfrac{du}{dx} - 4 = u^2$

$\dfrac{du}{dx} = u^2 + 4 \Rightarrow \int \dfrac{1}{u^2+4}\, du = \int dx$ (This is the variables separable part)

$\dfrac{1}{2}\tan^{-1}\left(\dfrac{u}{2}\right) = x + B$

$\dfrac{1}{2}\tan^{-1}\left(\dfrac{y+4x}{2}\right) = x + B$

$\tan^{-1}\left(\dfrac{y+4x}{2}\right) = 2x + C$ [$\because$ $C = 2B$]

$\dfrac{y+4x}{2} = \tan(2x + C) \Rightarrow y = 2\tan(2x + C) - 4x$ (shown)

2(d) $\dfrac{dy}{dx} = \dfrac{1-2y-4x}{1+y+2x} = \dfrac{1-2(y+2x)}{1+(y+2x)}$ --------(1)

Let $u = y + 2x$, then $\dfrac{du}{dx} = \dfrac{dy}{dx} + 2 \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 2$

Substituting these into (1) gives $\dfrac{du}{dx} - 2 = \dfrac{1-2u}{1+u} \rightarrow \dfrac{du}{dx} = \dfrac{1-2u}{1+u} + 2$

$= \dfrac{3}{1+u}$

$\int 1 + u\ du = 3\int dx \Rightarrow u + \dfrac{u^2}{2} = 3x + B$

$y + 2x + \dfrac{(y+2x)^2}{2} = 3x + B$

$2y + 4x + (y+2x)^2 = 6x + C$ [$\because$ $C = 2B$]

$\therefore$ $2y - 2x + (y+2x)^2 = C$ (shown)