Additional Complex Number Problems

1. The complex numbers z_1 and z_2 are given by $z_1 = 1 - ai$ and $z_2 = a - i$, where a is real and $-1 < a < 0$. Express $w = z_1 + z_2$ in the form $p + iq$, where p and q are real.

(i) Find $|w|$ in terms of a.

(ii) Find the exact value of $\arg(w)$.

(iii) In an Argand diagram, mark the points Z_1, Z_2 and W representing the complex Numbers z_1, z_2 and w respectively, showing clearly the geometrical relationship between the three points.

(iv) Point P represents the complex number $\frac{1}{w^n}$, where n is a positive integer, $n > 1$.

Find the smallest value of n such that the line passing through O and P is the perpendicular bisector of the line segment Z_1Z_2.

2 (a) The complex number w is such that $|w| = 2$ and $\arg(w) = \frac{\pi}{6}$. If the complex number z is given by $-1 + i$, find the modulus and argument of $\frac{w^{10}}{z^2}$. Hence or otherwise, find $\left| w - \frac{w^{10}}{z^2} \right|$.

(b) Let the complex number z be given by $a + ib$, where both a and b are real. Find the exact values of a and b, given that $\frac{1}{e^{iz}} = 2 + i$.

3(a) Given that $z = e^{\frac{\alpha}{2}i} + i$ where $-\pi < \alpha \le \pi$, find the exact value of α if $|z| = \sqrt{3}$.

(b) The complex number z_1 is given by $1 - 3i$.

(i) Find the modulus and argument of z_1, giving your answer in radians correct to 4 significant figures.

(ii) In an Argand diagram, the points S and T represent z_1 and z_1^* respectively.

Find the equation of the circle which passes through the origin, S and T in the form $|z-a|=r$, where a and r are real numbers.

(c) Given that the complex number z satisfies the equation of the circle in part b(ii), find the maximum value of $|z-3i|$.

4. Let $z=r(\cos\theta+i\sin\theta)$ be a complex number where $r>0,\ -\pi<\theta\le\pi$.

(i) Show that $\dfrac{z^2}{z^*}=r(\cos 3\theta+i\sin 3\theta)$.

(ii) If $z^2=iz^*$, find the value of r and the 3 values of θ.

5. On a single diagram, sketch the loci given by

(i) $|z-1|=1$, (ii) $\arg(z-2)=\dfrac{7\pi}{8}$.

Find the exact value of z satisfying both equations in (i) and (ii) and hence deduce the value of $\tan\dfrac{3\pi}{8}$.

6. Given that $z_1=2+i$ and $z_2=-1+2i$,

(i) Find the modulus and argument of each of the complex numbers z_1 and z_2.

(ii) In an Argand diagram with origin O, the complex numbers z_1, z_2 and z_1+z_2 are represented by the points A, B and C respectively. Show that OA is perpendicular to OB.

With the aid of a diagram, show that the angle between OC and the positive real axis is given by $\dfrac{\pi}{4}+\tan^{-1}\left(\dfrac{1}{2}\right)$.

(iii) By finding the argument of z_1+z_2, deduce that $\tan^{-1}3=\dfrac{\pi}{4}+\tan^{-1}\left(\dfrac{1}{2}\right)$.