Extreme Problem 4 Solutions

$$Z = 2^{\frac{1}{3}}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 2^{\frac{1}{3}} e^{-i\frac{\pi}{4}}$$

$$Z^3 = \left[2^{\frac{1}{3}} e^{-i\frac{\pi}{4}}\right]^3 = 2e^{-i\frac{3\pi}{4}} \quad \text{and} \quad Z^{-3} = \frac{1}{2e^{-j\frac{3\pi}{4}}} = \frac{1}{2}e^{i\frac{3\pi}{4}}$$

Hence, $Z^3 + \dfrac{1}{Z^3} = 2e^{-i\frac{3\pi}{4}} + \dfrac{1}{2}e^{i\frac{3\pi}{4}}$

$$= 2\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \frac{1}{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) + \frac{1}{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= -\sqrt{2} - \sqrt{2}i - \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}i$$

$$= \left(-\sqrt{2} - \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2\sqrt{2}} - \sqrt{2}\right)i$$

$$= \left(-\frac{5}{2\sqrt{2}}\right) + \left(-\frac{3}{2\sqrt{2}}\right)i$$

$$\left|Z^3 + \frac{1}{Z^3}\right| = \sqrt{\left(-\frac{5}{2\sqrt{2}}\right)^2 + \left(-\frac{3}{2\sqrt{2}}\right)^2} = \sqrt{\frac{34}{8}} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$$

$$\arg\left(Z^3 + \frac{1}{Z^3}\right) = -\left(\pi - \tan^{-1}\frac{3}{5}\right) = \tan^{-1}\frac{3}{5} - \pi$$

In Euler's form $Z^3 + \dfrac{1}{Z^3} = \dfrac{\sqrt{17}}{2} e^{(i)\left(\tan^{-1}\frac{3}{5} - \pi\right)}$

Recognising that $e^{\ln k} = k$ in general, where $k \in \Re$,

The above can be rewritten as

$$Z^3 + \frac{1}{Z^3} = e^{\ln\left(\frac{\sqrt{17}}{2}\right)} e^{(i)\left(\tan^{-1}\frac{3}{5} - \pi\right)} = e^{\ln\left(\frac{\sqrt{17}}{2}\right) + (i)\left(\tan^{-1}\frac{3}{5} - \pi\right)}$$

By comparison, $a = \ln\frac{\sqrt{17}}{2}$ and $b = \tan^{-1}\frac{3}{5} - \pi$ (shown)